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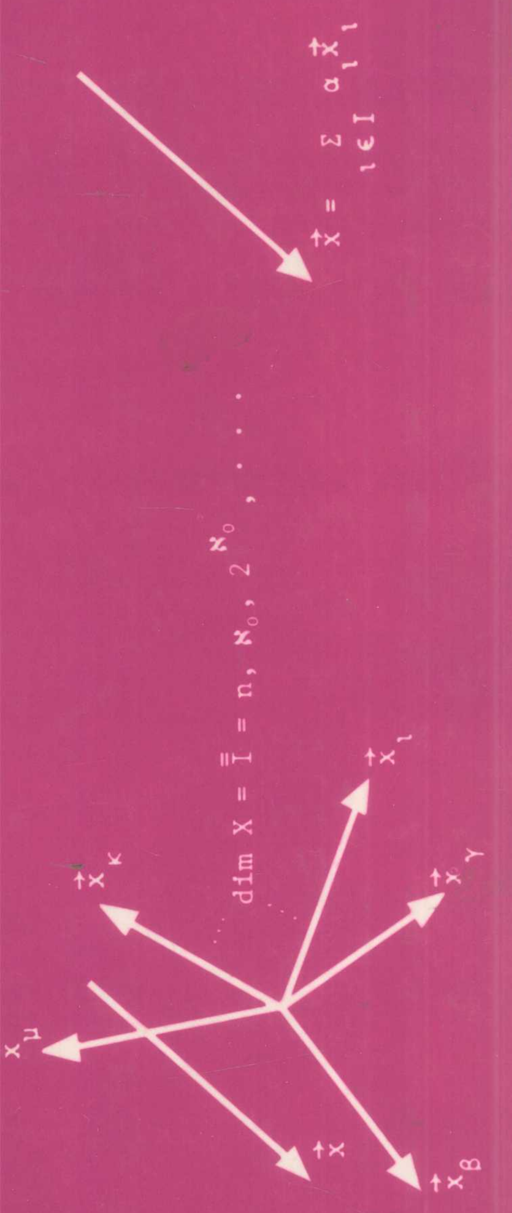

FINITE AND INFINITE DIMENSIONAL LINEAR SPACES

Richard D. Järvinen

FINITE AND INFINITE DIMENSIONAL LINEAR SPACES

A Comparative Study in Algebraic and Analytic Setting

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FOREWORD

This volume gives an accessible account of the fundamental concepts of the theory of linear spaces with finite dimension and the theory of linear spaces with infinite dimension, with special emphasis upon their similarities and the contrasts between them.

The ideas and methods in this work are applied to a careful generalization of finite systems of linear algebraic equations to infinite systems in perspective with the theory of linear operators. Some theorems dealing with infinite matrices and infinite systems of algebraic equations are included.

The book concludes with some of the most important unsolved problems in the theory of linear spaces. Let us hope that this list contributes to the spread of mathematical richness among our young generation with the consequence that many of the problems will be attacked and brought to solution.

I find this to be an excellent book, suitable as a reference or as a text. It is clear, very well written, and carefully organized.

University of California at Berkeley — Themistocles M. Rassias

PREFACE

This book was written to compare the theory of linear spaces with finite dimension and the theory of linear spaces with infinite dimension. It does so by showing contrasts and similarities. First, comparisons are made when the spaces are not assumed to have an accompanying topology--a strictly algebraic, nonanalytic setting. Second, comparisons are made when the spaces are endowed with a topology, the topological structure arising from an inner product, a norm, a metric, or a nonmetric assumption.

A feature of this book, while not directly related to the main objective, is that the exposition contains extensive lines of development, in both the finite and infinite dimensional theory of linear spaces, where topological as well as nontopological conditions are included. These "lines" unravel as the comparative study is pursued. We have included a comprehensive collection of theorems each having a principal role in the overall theory of linear spaces. In light of the results found in this volume, the book can serve as a reference for both students and accomplished mathematicians and scientists.

The book is suitable as a text as well. The order in which theorems are presented generally permits one to prove each theorem by relying upon previously established results. This is strongly the case for the first chapter of the book where for nearly all of the theorems in the infinite dimensional theory a proof is provided. Numerous problems covering a broad scope of ideas are posed at the end of the first three chapters to assist the reader in expanding or testing his or her understanding of the material. A special fourth chapter is devoted to significant research questions in the field of this monograph. While the primary audience for a thorough study of the first three chapters is likely to be second-year or third-year graduate students of mathematics, the first chapter has been used with undergraduate mathematics majors for independent study courses and as a textbook for a seminar course on infinite dimensional linear spaces.

The following is the method by which the purposes of this book are realized. A theorem central to the subject of linear spaces with finite dimension is stated. It is accompanied by an analogous theorem from the theory of linear spaces with infinite dimension when an appropriate analogue exists. If a theorem in the finite dimensional theory has no "reasonable" infinite dimensional analogue, an illustrative example is presented or cited to bear this out. The analogue of a theorem from the finite dimensional theory that results simply by changing the condition of finite dimensionality to one of infinite dimensionality is, let us agree, the "strongest analogue."

An overview of much of mathematics can be derived from a simple but thoughtful reading of these pages. One will find a representative blend of variously labeled branches of mathematics. Here is a meeting place for the foundations of mathematics, classical analysis, linear algebra, abstract algebra, and topology. One will find here pure and applied mathematics, however one might choose to separate those realms, if at all. And if we might single out one essential group for whom a knowledge of this mathematics should prove to be particularly valuable, we cite those who provide undergraduate instruction. This exposition can be gainfully used to develop a mathematical perspective upon the first four or five years of a traditional college or university mathematics education.

Additional features of this book include a careful distinction and treatment of the notions of algebraic and analytic bases for linear spaces, the generalization of finite systems of linear algebraic equations to infinite systems, to integral equations, and finally to linear operators generally, and the inclusion of some theorems dealing with infinite matrices and infinite systems of algebraic equations taken from a variety of sources and not known to be found collectively in one volume.

The first chapter can be used as text material for a course in the algebraic theory of infinite dimensional linear spaces. It can also serve as a skeletal exposition of the analogous theory for finite dimensional spaces. But with only occasional exceptions, the proofs for the latter theory are not found here. However, students can be assigned the task, and are encouraged to take it upon themselves, to establish the companion finite dimensional case results.

The subject of linear spaces with infinite dimension was developed mainly during this century. Its origin is due largely to Ivar Fredholm (1866-1927) and Vito Volterra (1860-1940). Each of these mathematicians visualized the limiting case of the finite system of linear algebraic equations as the number of equations and unknowns become infinite, as leading to the theory of integral equations. In turn, the theory of integral equations, substantially influenced by the work of David Hilbert (1861-1943), John von Neumann (1903-1957), Erhard Schmidt (1876-1959), and Frigyes Riesz (1880-1956), gave rise to the theory of linear spaces with infinite dimension. Hilbert space and Banach space are prime examples.

Regarding some conventions that are made in this presentation, in Chapter 1 a theorem in the theory of linear spaces with finite dimension is assigned a letter of the English alphabet, e. g. , Theorem 1.3-D, the latter designating a theorem from Chapter 1, Section 3. The infinite dimensional analogue or appropriate accompanying example is assigned the same letter except that the symbol "prime" is attached, e. g., Theorem 1.3-D'. Note that we use exclusively the term "linear space" when referring to what some authors in certain settings might call a "vector space" while in other circumstances might use the term "linear space." One can make a good case for using either of the terms as well as "linear vector space." The structure of these spaces is linear; the objects in the spaces are vectors.

I take this opportunity to express my sincere gratitude to Professors Erik Hemmingsen, Donald Kibbey, and especially Jerome Blackman, all of Syracuse University, for their varied and timely assistance in the development of this manuscript. A first version of this exposition was written by the author as a doctoral dissertation under the supervision of Dr. Blackman, the idea of that monograph being to expose an area of mathematics in which some of the original work of the author would hold a place rather than the other way around. I owe these mathematicians more than I will ever be able to give in return. The author is greatly indebted to Themistocles Rassias of Greece who read the entire manuscript and contributed many of the exercises and the bulk of the research problems. His inspiring manner and helpful conversation, especially at the University of California at Berkeley during the summer of 1977 at which time the author was a Visiting Scholar at Berkeley, will never be forgotten.

Special appreciation is extended to Hjalmer Anderson and to John O'Dougherty for their influence during the early education of the author in the subjects of mathematics and language, respectively, and to my parents, who continued to teach me the value of education, beginning at a point earlier in my life than I am able to recall. Lastly, I thank my wife and sons for their patience and support.

The study of infinite dimensional linear spaces is currently an active area of research in mathematics. This book is, of course, but a brief survey of the theory of linear spaces, in keeping with the intentions already mentioned. Dr. Rassias has called it "a book of ideas." The hope of the author is that these pages will be instrumental in the creation of new ideas in the expanding world of mathematics and that what is written here will help to make known the nature of this branch of mathematics.

Richard D. Järvinen

L'essence des mathématiques c'est la liberté.

G. Cantor

It is difficult to give an idea of the vast extent of modern mathematics.

A. Cayley

There's lots of room left in Hilbert space.

S. MacLane

CONTENTS

Chapter 1

NONTOPOLOGICAL LINEAR SPACE THEORY

The setting in this chapter is exclusively algebraic, nonanalytic. Convergence questions are not encountered. We present theorems that are central to the subject of finite dimensional linear spaces, theorems that are found in thorough treatments of linear algebra or finite dimensional linear space theory. We ask whether or not such theorems remain true if the condition of finite dimensionality is lifted and replaced by one of infinite dimensionality. If there is no reasonable analogue, an example is provided to show why.

We assume that the reader is able to handle the mathematical sophistication that is proper to a careful treatment of the notion of function and some of its properties. Only inasmuch as we have need for them, we present results from the foundations of mathematics, set theoretic notions that enable us to do the "transfinite trickery"--a term used by Paul Halmos (1958)--to prove the theorems about infinite dimensional linear spaces that appear. It is not our intention to prove the finite dimensional case theorems, although some of them are proved in the process of establishing the infinite dimensional case results.

§1. THE CONCEPTS OF GROUP, FIELD, AND LINEAR SPACE

The notion of function (map, mapping, transformation, operator, correspondence) has been an evolving one. In fact, for most of the last century this concept was not understood in the same sense that it is today. What is clear, however, is that this notion in its various evolutionary stages has played a key role in the development of mathematics. Progress in the field of mathematical analysis, for instance, has depended critically upon advantageous use of refinements of this concept. A careful exposition of this dependence together with a thorough treatment of the historical role of the notion of a function is found in Manheim (1964).

As the function concept is understood today [see Hewitt and Stromberg (1965) for a full and careful treatment of this notion] it holds a primary position in mathematics. In the field of algebra, for example, the function concept is usually at the base of the definitions that form the underpinnings of an algebraic system. This is the case for the definitions of group, field, and linear space. For, modulo some equivalence relations, one function is a group, two constitute a field, and four define a linear space. In fact an arbitrary vector in an arbitrary linear space can be realized as a function, an important observation that is made in subsequent pages. Needless to say, the functions we have referred to must possess some rather special properties, and these are detailed in the following paragraphs.

To be definite as well as brief, we define a function, say f, to be a collection of ordered pairs for which $(x, y) \in f$ and $(x, z) \in f$ imply $y = z$.

Assume Γ_1, Γ_2, Γ_3 are three nonempty, possibly pairwise distinct sets. Let $(\gamma_1, \gamma_2) \to \gamma_1 * \gamma_2$ denote a mapping of $\Gamma_1 \times \Gamma_2$ into Γ_3. We call this type of function a <u>pairing</u>, or a $*$-<u>pairing</u> (in light of the notation), of Γ_1 and Γ_2 into Γ_3 and refer to $\gamma_1 * \gamma_2$ as the $*$-<u>join</u> of γ_1 and γ_2. When no ambiguity is possible, we shall say "pairing" rather than "$*$-pairing." Pairings are precisely the functions needed in making the definitions of group, field, and linear space.

It should be noted that in the last paragraph we have tacitly assumed that a notion of equality, i. e. , an equivalence relation, has been defined on each of the sets $\Gamma_1 \times \Gamma_2$ and Γ_3, for otherwise it could not be decided whether or not a given object in the domain has a unique image under the pairing. If we assume that ordered pairs in $\Gamma_1 \times \Gamma_2$ are equal if and only if their respective coordinates are equal, then we arrive at a minimal mathematical structure within which it can be concluded that equals joined to equals are equal, a result so commonly used in establishing general properties of groups, fields, linear spaces, and other algebraic systems.

A $*$-pairing of $\Gamma_1 \times \Gamma_2$ into Γ_3 with $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma$ is denoted $(\Gamma, *)$ and is called a commutative group provided

1. $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ for all $\alpha, \beta, \gamma \in \Gamma$.
2. There exists (identity) $e \in \Gamma$ such that $e * \alpha = \alpha$ for all $\alpha \in \Gamma$.
3. For each $\alpha \in \Gamma$ there exists (inverse) $\alpha^{-1} \in \Gamma$ such that $\alpha^{-1} * \alpha = e$.
4. $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in \Gamma$.

It is believed that Leopold Kronecker (1823-1891) in 1870 gave the earliest explicit set of postulates for an abstract group.

Suppose $(\Gamma, *)$ is a commutative group and $(\Gamma - \{e\}, \hat{*})$ is also a commutative group with identity $\hat{e}$. Suppose further that $*$-joins and $\hat{*}$-joins are related by

5. $\alpha \mathbin{\hat{*}} (\beta * \gamma) = (\alpha \mathbin{\hat{*}} \beta) * (\alpha \mathbin{\hat{*}} \gamma)$ for all $\alpha, \beta, \gamma \in \Gamma$.
6. $(\alpha * \beta) \mathbin{\hat{*}} \gamma = (\alpha \mathbin{\hat{*}} \gamma) * (\beta \mathbin{\hat{*}} \gamma)$ for all $\alpha, \beta, \gamma \in \Gamma$.

Then the commutative groups $(\Gamma, *)$ and $(\Gamma - \{e\}, \hat{*})$ together with properties 5 and 6 is called a field and is denoted $[(\Gamma, *), (\Gamma, \hat{*})]$.

One should check to see if properties 1, 2, 3, and 4 continue to hold when they are stated for the $\hat{*}$-pairing involving e, for little--

just 5 and 6--has been assumed about $\hat{*}$-joins which involve e. It is implicit in the distributive laws 5 and 6 that $\alpha \hat{*} e$ and $e \hat{*} \alpha$ are defined for all $\alpha \in \Gamma$. The same laws enable us to deduce that $\alpha \hat{*} e = e$ and $e \hat{*} \alpha = e$ for all $\alpha \in \Gamma$, for from 5, $\alpha \hat{*} e = \alpha \hat{*} (e * e) = (\alpha \hat{*} e) * (\alpha \hat{*} e)$, and thus $e = \alpha \hat{*} e$. Analogously one can show $e \hat{*} \alpha = e$.

The following are consequences of this definition of a field, while they are often found as integral parts of equivalent definitions of a field:

(a) The set Γ has at least two members.

(b) The identities e and $\hat{e}$ are distinct.

(c) The identity e has no inverse with respect to the $\hat{*}$-pairing.

The first of the above follows since $(\Gamma, *)$ and $(\Gamma, \hat{*})$ are distinct (nonempty) functions, the second because $(\Gamma, *)$ and $(\Gamma - \{e\}, \hat{*})$ are distinct groups, and the third because $e \hat{*} \alpha$ and $\alpha \hat{*} e$, each now seen to be equal to e for all $\alpha \in \Gamma$, are certainly different from $\hat{e}$.

Suppose $[(\Gamma, *), (\Gamma, \hat{*})]$ is a field and $(X, \check{*})$ is a commutative group. Further, assume $\tilde{*} : \Gamma \times X \to X$ is a pairing and that the $*$-, $\hat{*}$-, $\check{*}$-, and $\tilde{*}$-joins are related by

7. $\alpha \tilde{*} (\beta \tilde{*} x) = (\alpha \hat{*} \beta) \tilde{*} \vec{x}$ for all $\alpha, \beta \in \Gamma$ and $\vec{x} \in X$.

8. $\hat{e} \tilde{*} \vec{x} = \vec{x}$ for all $\vec{x} \in X$.

9. $\alpha \tilde{*} (\vec{x} \check{*} \vec{y}) = (\alpha \tilde{*} \vec{x}) \check{*} (\alpha \tilde{*} \vec{y})$ for all $\alpha \in \Gamma$ and all $\vec{x}, \vec{y} \in X$.

10. $(\alpha * \beta) \tilde{*} \vec{x} = (\alpha \tilde{*} \vec{x}) \check{*} (\beta \tilde{*} \vec{x})$ for all $\alpha, \beta \in \Gamma$ and all $\vec{x} \in X$.

Together these four special functions constitute a <u>linear (vector) space over the field</u> Γ. More pointedly we have defined what is sometimes called a <u>left vector space over</u> Γ. Analogously we can define a <u>right vector space over</u> Γ by replacing $\tilde{*} : \Gamma \times X \to X$ with $\tilde{*} : X \times \Gamma \to X$. But then the left and right linear spaces are algebraically equivalent, i. e. , isomorphic, and there is no need to distinguish them. Each object in X is called a <u>vector</u> while each element of Γ is called a <u>scalar</u>.

Because of the clarity context brings, there is no need to continue with the cumbersome notation which has at this point accumulated. We replace the symbols $*$ and $\check{*}$ by the symbol +, the symbols $\hat{*}$ and $\tilde{*}$ by $\cdot$ or no symbol at all, and call the first notation <u>field addition</u> or <u>vector addition</u>, as context dictates, and the second notation either <u>field multiplication</u> or <u>scalar multiplication</u>. Instead of saying "let $[(\Gamma, *), (\Gamma, \hat{*}), (X, \check{*}), (\Gamma \times X, \tilde{*}, X)]$ be a linear space," we simply shall say "let X be a linear space over the field Γ," or even more briefly, "let X be a linear space" when it is clear what the field is.

Later we refer to the following notion. A field Γ is said to have <u>characteristic</u> n if n is the least positive integer for which $n\gamma = e$ for for every $\gamma \in \Gamma$; if no such n exists, then Γ is said to have characteristic 0. The notation $n\gamma$ means γ joined to itself n times. The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are examples of fields of characteristic zero. It is customary to have the letter F represent a field, and we adopt this convention.

§2. SET THEORETIC NOTIONS AND OTHER PRELIMINARIES

In various branches of mathematics circumstances present themselves in which the methods of elementary set theory are not adequate to permit constructions, proofs, or definitions to be formulated that may be needed to develop a theory or model. In 1904 Ernst Zermelo (1871-1953) stated an axiom different from those found in elementary set theory, called the axiom of choice. It permits one to make constructions, proofs, and definitions beyond those elementary set theory allows. Some of these appear in the next section.

During the early 1960s, Paul J. Cohen (1934-) proved that the axiom of choice is independent of the other axioms of set theory. That result allows the development of a consistent mathematical system in which the axiom of choice can be either