

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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W.A. Light
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Approximation Theory
in Tensor Product Spaces



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Mathematics Subject Classification (1980): Primary: 41A63, 41A65
Secondary: 41-02, 41A30, 41A45, 41A50

ISBN 3-540-16057-4 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-16057-4 Springer-Verlag New York Heidelberg Berlin Tokyo

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© by Springer-Verlag Berlin Heidelberg 1985
Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

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DEDICATION

This work is dedicated to the memory of

Robert Schatten

(1911 - 1977)

who did much of the pioneering work in
the theory of tensor products of Banach spaces.

PREFACE

In the past two decades, a new branch of approximation theory has emerged; it concerns the approximation of multivariate functions by combinations of univariate ones. The setting for these approximation problems is often a Banach space which is the tensor product of two or more simpler spaces. Approximations are usually sought in subspaces which are themselves tensor products. While these are infinite dimensional, they may share some of the characteristics of finite-dimensional subspaces. The usual questions from classical approximation theory can be posed for these approximating subspaces, such as (i) Do best approximations exist? (ii) Are best approximations unique? (iii) How are best approximations characterized? (iv) What algorithms can be devised for computing best approximations? (v) Do there exist simple procedures which provide "good" approximations, in contrast to "best" approximations? (vi) What are the projections of least norm on these subspaces? and (vii) what are the projection constants of these subspaces?

This volume surveys only a part of this growing field of research. Its purpose is twofold: first, to provide a coherent account of some recent results; and second, to give an exposition of the subject for those not already familiar with it. We cater for the needs of this latter category of reader by adopting a deliberately slow pace and by including virtually all details in the proofs. We hope that the book will be useful to students of approximation theory in courses and seminars.

Expert readers may wish to omit a reading of the first chapter, which gives an introduction to the tensor product theory of Banach spaces. The material on approximation theory occupies the next eight chapters. Results needed in proofs but perhaps not familiar to every reader are collected in two appendices (Chapters 10 and 11). Finally, there are historical notes and a large collection of references, some of which are only peripheral to our theme.

Notation and conventions are standard throughout, and we often do not stop to define notation which we expect to be familiar. However, a table of notation has been placed just before the index.

We are glad to be able to thank a number of colleagues for pleasant collaboration over

the years on matters relating to these notes: Carlo Franchetti (Florence), Manfred von Golitschek (Würzburg), Julie Halton (Lancaster), Sue Holland (Lancaster), John Respass (Austin), and Lin Sulley (Lancaster and Ipswich).

During the preparation of the manuscript, the second author was supported by grants from the University of Texas and the Science and Engineering Research Council of Great Britain. For these grants, and for the hospitality of the University of Lancaster, he is deeply grateful.

We are very much indebted to Ms. Jan Duffy of the University of Texas Mathematics Department, who undertook the arduous task of rendering our manuscript into a computer file for processing by the TEX typesetting system. The pleasing appearance which (we think) the book possesses is due entirely to the skill and good judgement of Ms. Duffy.

W.A. Light E.W. Cheney

Lancaster, July 1985

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CHAPTER 1

AN INTRODUCTION TO TENSOR PRODUCT SPACES

The purpose of this chapter is to introduce some of the basic theory of the tensor product of two Banach spaces. All of this material can be found in other sources, but the treatment here is particularly designed to meet the needs of the subsequent chapters. As mentioned in the preface, we have tried to expound the subject in a way that leaves very few arguments for the reader to supply.

There are two sources of information on this topic which we should mention at the outset. One is Schatten's monograph [154], which gives a very careful treatment of the foundations of the subject. The other is the survey of vector measures by Diestel and Uhl [55]. Chapter 8 of [55] provides a brief introduction to tensor products of Banach spaces and covers many recent results in this area.

Let X and Y be Banach spaces, and denote their duals by X^* and Y^* , respectively. We shall construct formal expressions $\sum_{i=1}^n x_i \otimes y_i$ where $x_i \in X$, $y_i \in Y$ and $n \in \mathbb{N}$. We will regard such an expression as defining an operator $A : X^* \rightarrow Y$, given by

$$A\phi = \sum_{i=1}^n \phi(x_i)y_i \quad (\phi \in X^*).$$

Amongst all these formal expressions we introduce the relation

$$\sum_{i=1}^n x_i \otimes y_i \sim \sum_{i=1}^m a_i \otimes b_i$$

if both expressions define the same operator from X^* to Y . This is clearly an equivalence relation on the set of all such formal expressions. We shall henceforward be interested only in the equivalence classes of this relation, and will denote the set of all such equivalence classes by $X \otimes Y$. We shall abuse notation in the usual way by referring to the expression $\sum_{i=1}^n x_i \otimes y_i$ as a member of $X \otimes Y$ when we intend to refer to the equivalence class of expressions containing $\sum_{i=1}^n x_i \otimes y_i$. For any $\alpha \in \mathbb{R}$ we define multiples of (the equivalence class of expressions containing) $\sum_{i=1}^n x_i \otimes y_i$ by (the equivalence class of expressions containing) $\sum_{i=1}^n \alpha x_i \otimes y_i$. Similarly, we define addition by

$$\sum_{i=1}^n x_i \otimes y_i + \sum_{i=n+1}^m x_i \otimes y_i = \sum_{i=1}^m x_i \otimes y_i.$$

All algebraic identities in $X \otimes Y$ are based upon the interpretation of expressions as linear operators. Thus one easily verifies such identities as

$$\begin{aligned}x \otimes (u + v) &= x \otimes u + x \otimes v \\ \alpha x \otimes y &= x \otimes \alpha y \\ x \otimes 0 &= 0 \otimes 0.\end{aligned}$$

Two observations are perhaps helpful at this juncture. Firstly, the symbol “+” is being used only as a separator in our formal expression $\sum_{i=1}^n x_i \otimes y_i$ and in our definition of the addition of two such expressions. Secondly, the construction thus far makes no use of the topological structure of X or Y . Thus we could have begun with linear spaces and algebraic duals rather than Banach spaces. If we begin with Banach spaces and identify $\sum_{i=1}^n x_i \otimes y_i$ with an operator A mapping the algebraic dual of X into Y then the equivalence classes that make up $X \otimes Y$ remain the same.

It is clear that scalar multiplication of $\sum_{i=1}^n x_i \otimes y_i$ by α is equivalent to multiplying the associated operator A by α , and that the addition of this expression to $\sum_{j=1}^m a_j \otimes b_j$ is equivalent to adding A to the operator B associated with this last expression. With these definitions $X \otimes Y$ forms a linear space, called the **algebraic tensor product**.

1.1 LEMMA. *Every expression $\sum_{i=1}^n x_i \otimes y_i$ is equivalent to either $0 \otimes 0$ or to an expression $\sum_{i=1}^m a_i \otimes b_i$ where $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$ are linearly independent sets.*

PROOF. Suppose that for instance $x_n = \sum_{j=1}^{n-1} \alpha_j x_j$. Then $\sum_{i=1}^n x_i \otimes y_i$ defines the operator $A : X^* \rightarrow Y$ where, for $\phi \in X^*$,

$$\begin{aligned}A\phi &= \sum_{i=1}^n \phi(x_i)y_i = \sum_{i=1}^{n-1} \phi(x_i)y_i + \phi(x_n)y_n \\ &= \sum_{i=1}^{n-1} \phi(x_i)y_i + \phi\left(\sum_{j=1}^{n-1} \alpha_j x_j\right)y_n \\ &= \sum_{i=1}^{n-1} \phi(x_i)y_i + \sum_{j=1}^{n-1} \alpha_j \phi(x_j)y_n \\ &= \sum_{i=1}^{n-1} \phi(x_i)(y_i + \alpha_i y_n).\end{aligned}$$

Hence $\sum_{i=1}^n x_i \otimes y_i$ has the representation $\sum_{i=1}^{n-1} a_i \otimes b_i$ where $a_i = x_i$ and $b_i = y_i + \alpha_i y_n$. We may repeat this process until we arrive at either a representation in which $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$ are linearly independent or one of the representations $x \otimes 0$ or $0 \otimes y$, which are both equivalent to $0 \otimes 0$. ■

The space $X \otimes Y$ is generated by elements of the form $x \otimes y$, which are called **dyads**. This observation is often used to simplify linear arguments.

From the preceding discussions it is clear that $X \otimes Y$ may be regarded as a subspace of the space of continuous linear operators of finite rank from X^* into Y . Usually this subspace is a proper one. However, in the case when X is reflexive every continuous finite rank linear operator from X^* into Y can be identified with an expression $\sum_{i=1}^n x_i \otimes y_i$ as follows. Suppose A is such an operator with range determined by linearly independent $b_1, \dots, b_n$. Then for $\phi \in X^*$,

$$A\phi = \sum_{i=1}^n \alpha_i(\phi) b_i \quad \text{where} \quad \alpha_i(\phi) \in \mathbb{R}.$$

Standard arguments show that the α_i are in X^{**} and hence in X , whence $\alpha_i(\phi) = \phi(a_i)$ for suitable $a_i \in X$, $1 \leq i \leq n$. Hence,

$$A\phi = \sum_{i=1}^n \phi(a_i) b_i$$

and A is associated with $\sum_{i=1}^n a_i \otimes b_i$.

It is possible to construct various norms on $X \otimes Y$ using the norms in X and Y . The most obvious way to introduce a norm which is independent of the representation of the equivalence class is to assign to $\sum_{i=1}^n x_i \otimes y_i$ the norm it receives when regarded as an operator from X^* to Y ; viz.,

$$\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup \left\{ \left\| \sum_{i=1}^n \phi(x_i) y_i \right\| : \phi \in X^*, \|\phi\| = 1 \right\}.$$

Notice that for a dyad $x \otimes y$ we have

$$\begin{aligned} \lambda(x \otimes y) &= \sup \{ \|\phi(x)y\| : \phi \in X^*, \|\phi\| = 1 \} \\ &= \sup \{ |\phi(x)| \|y\| : \phi \in X^*, \|\phi\| = 1 \} \\ &= \|x\| \|y\|. \end{aligned}$$

Such a norm on $X \otimes Y$, for which the norm of a dyad equals the product of the norms of its two components, is termed a **crossnorm**. Given any two Banach spaces X and Y there is a rich supply of crossnorms on $X \otimes Y$.

We can form a second linear space from X and Y by considering $X^* \otimes Y^*$. This consists of all expressions

$$\sum_{i=1}^n \phi_i \otimes \psi_i \quad \text{where} \quad \phi_i \in X^* \quad \text{and} \quad \psi_i \in Y^*.$$

In addition to other interpretations such an expression may be considered as a linear form on $X \otimes Y$ by defining

$$\left(\sum_{i=1}^n \phi_i \otimes \psi_i \right) \left(\sum_{j=1}^m x_j \otimes y_j \right) = \sum_{i=1}^n \sum_{j=1}^m \phi_i(x_j) \psi_i(y_j).$$

One verifies easily that this definition is *proper*; that is, it is invariant over the equivalence classes. Note that it would have sufficed to give the above definition for a pair of dyads:

$$(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y).$$

1.2 DEFINITION. Let α be a norm on $X \otimes Y$. We say that α is a *crossnorm* if, for all $x \in X$ and $y \in Y$,

$$\alpha(x \otimes y) = \|x\| \|y\|.$$

We say that α is a *reasonable norm* if, for all $\phi \in X^*$ and $\psi \in Y^*$, the linear form $\phi \otimes \psi$ is bounded on $(X \otimes Y, \alpha)$ and has norm equal to $\|\phi\| \|\psi\|$.

1.3 DEFINITION. Let α be a norm on $X \otimes Y$. Define α^* on $X^* \otimes Y^*$ by the equation

$$\alpha^* \left(\sum_{i=1}^n \phi_i \otimes \psi_i \right) = \sup \left\{ \sum_{i=1}^n \sum_{j=1}^m \phi_i(x_j) \psi_i(y_j) : \alpha \left(\sum_{j=1}^m x_j \otimes y_j \right) = 1 \right\}.$$

We admit the possibility that α^* may take $+\infty$ as a value.

Observe that if α is reasonable then α^* is the norm induced on $X^* \otimes Y^*$ by considering the latter as a linear subspace of $(X \otimes Y, \alpha)^*$. The norm α^* is the *associate* of α .

1.4 LEMMA. Let α be a norm on $X \otimes Y$ satisfying

- (i) $\alpha(x \otimes y) \leq \|x\| \|y\|$ for all $x \in X$ and $y \in Y$;
- (ii) $\alpha^*(\phi \otimes \psi) \leq \|\phi\| \|\psi\|$ for all $\phi \in X^*$ and $\psi \in Y^*$.

Then α is a *reasonable crossnorm*.

PROOF. If $x \in X$, $y \in Y$, $\phi \in X^*$, and $\psi \in Y^*$, then

$$\phi(x)\psi(y) = (\phi \otimes \psi)(x \otimes y) \leq \alpha^*(\phi \otimes \psi)\alpha(x \otimes y) \leq \begin{cases} \|\phi\| \|\psi\| \alpha(x \otimes y) \\ \|x\| \|y\| \alpha^*(\phi \otimes \psi). \end{cases}$$

In this inequality, take a supremum as ϕ and ψ range over the unit cells of X^* and Y^* , getting

$$\|x\| \|y\| \leq \alpha(x \otimes y).$$

If we take a supremum as x and y range over the unit cells of X and Y , we have

$$\|\phi\| \|\psi\| \leq \alpha^*(\phi \otimes \psi). \quad \blacksquare$$

1.5 LEMMA. *Suppose that α is a reasonable crossnorm on $X \otimes Y$. Then α^* is a reasonable crossnorm on $X^* \otimes Y^*$.*

PROOF. Since α is a reasonable norm, α^* is a crossnorm. In order to show that α^* is a reasonable norm, take $\mu \in X^{**}$ and $\nu \in Y^{**}$. By 1.4 we need only show that

$$\alpha^{**}(\mu \otimes \nu) \leq \|\mu\| \|\nu\|.$$

By Goldstine's theorem [57, p.424] there exist nets

$$\{x_\beta\} \subset X, \quad \{y_\gamma\} \subset Y$$

such that μ and ν are the weak* limits of $\{x_\beta\}$ and $\{y_\gamma\}$, and

$$\|x_\beta\| \leq \|\mu\|, \quad \|y_\gamma\| \leq \|\nu\|.$$

Let

$$\theta = \sum_{i=1}^n \phi_i \otimes \psi_i \in X^* \otimes Y^*.$$

Then

$$|\theta(x_\beta \otimes y_\gamma)| \leq \alpha^*(\theta) \alpha(x_\beta \otimes y_\gamma) \leq \alpha^*(\theta) \|x_\beta\| \|y_\gamma\| \leq \alpha^*(\theta) \|\mu\| \|\nu\|.$$

By taking a limit in β and γ we get

$$\begin{aligned} |\theta(\mu \otimes \nu)| &\leq \alpha^*(\theta) \|\mu\| \|\nu\| \\ \text{or } |(\mu \otimes \nu)(\theta)| &\leq \alpha^*(\theta) \|\mu\| \|\nu\|. \end{aligned}$$

This shows that $\alpha^{**}(\mu \otimes \nu) \leq \|\mu\| \|\nu\|$. $\blacksquare$

Returning now to our only example thus far of a crossnorm, we refer to λ as the least of the reasonable crossnorms by virtue of the following lemma.

1.6 LEMMA. *The norm λ is a reasonable crossnorm on $X \otimes Y$. Moreover, if α is any reasonable crossnorm on $X \otimes Y$ then*

$$\lambda(z) \leq \alpha(z) \quad \text{for all } z \in X \otimes Y.$$

PROOF. In our preliminary discussion of λ we verified the crossnorm property. We have

$$(1) \quad \lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup \left\{ \left\| \sum_{i=1}^n \phi(x_i) y_i \right\| : \phi \in X^*, \|\phi\| = 1 \right\}.$$

If now $\phi \in X^*$, $\psi \in Y^*$ and z is any element of $X \otimes Y$ then by Eq. (1)

$$\begin{aligned} |(\phi \otimes \psi)(z)| &= \|\phi\| \|\psi\| \left| \left(\frac{\phi}{\|\phi\|} \otimes \frac{\psi}{\|\psi\|} \right)(z) \right| \\ &\leq \|\phi\| \|\psi\| \lambda(z). \end{aligned}$$

Thus $\lambda^*(\phi \otimes \psi) \leq \|\phi\| \|\psi\|$ and so by 1.4 λ is a reasonable crossnorm.

Now let α be a reasonable crossnorm on $X \otimes Y$. Let $z \in X \otimes Y$, $\phi \in X^*$ and $\psi \in Y^*$. Then

$$|(\phi \otimes \psi)(z)| \leq \alpha^*(\phi \otimes \psi)\alpha(z) = \|\phi\| \|\psi\| \alpha(z).$$

Taking a supremum for ϕ, ψ belonging to the unit cells of X^* and Y^* and using Eq. (1) gives $\lambda(z) \leq \alpha(z)$. ■

It is also possible to define the greatest crossnorm. We do this next and then proceed to investigate each of these norms in somewhat greater detail.

1.7 DEFINITION. For any $z \in X \otimes Y$ we define

$$\gamma(z) = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : x_i \in X, y_i \in Y, z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

1.8 LEMMA. The norm γ is a reasonable crossnorm on $X \otimes Y$. Furthermore, if α is any crossnorm on $X \otimes Y$ then $\gamma(z) \geq \alpha(z)$ for all $z \in X \otimes Y$.

PROOF. The first issue at stake here is whether γ is a norm at all. We check first that if $z \neq 0$ then $\gamma(z) > 0$. Let

$$z = \sum_{i=1}^n x_i \otimes y_i.$$

If z is nonzero then the operator L_z defined by z is not the zero operator from X^* into Y . Hence there exists a $\phi \in X^*$ such that $\|\phi\| = 1$ and $L_z \phi \neq 0$. Thus

$$0 < \|L_z \phi\| \leq \left\| \sum_{i=1}^n \phi(x_i) y_i \right\| \leq \sum_{i=1}^n |\phi(x_i)| \|y_i\| \leq \sum_{i=1}^n \|x_i\| \|y_i\|.$$

Since this holds for all representations of z we have $\gamma(z) > 0$. The definition of $\gamma(z)$ is clearly independent of the representation of $z \in X \otimes Y$ and the remaining properties of a norm are easily verified.

To establish that γ is a crossnorm, suppose that $x \otimes y$ and $\sum_{i=1}^n x_i \otimes y_i$ are two equivalent expressions for $z \in X \otimes Y$. Then for all $\phi \in X^*$ we have

$$\phi(x)y = \sum_{i=1}^n \phi(x_i)y_i.$$