An Introduction to General Relativity

L.P. Hughston and K.P. Tod

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Preface

Omnia profecto cum se coelestibus rebus referet ad humanas, excelsius magnificentiusque, et dicus et sentiet. (*The contemplation of celestial things will make a man both speak and think more sublimely and magnificently when he descends to human affairs.*)

—Cicero

IT IS INEVITABLE that with the passage of time Einstein's general relativity theory, his theory of gravitation, will be taught more frequently at an undergraduate level. It is a difficult theory—but just as some athletic records fifty years ago might have been deemed nearly impossible to achieve, and today will be surpassed regularly by well-trained university sportsmen, likewise Einstein's theory, now over seventy-five years since creation, is after a lengthy gestation making its way into the world of undergraduate mathematics and physics courses, and finding a more or less permanent place in the syllabus of such courses. The theory can now be considered both an accessible and a worthy, serious object of study by mathematics and physics students alike who may be rather above average in their aptitude for these subjects, but who are not necessarily proposing, say, to embark on an academic career in the mathematical sciences. This is an excellent state of affairs, and can be regarded, perhaps, as yet another aspect of the overall success of the theory.

But the study of general relativity at an undergraduate level does present some special problems. First of all, the content of the course must be reasonably well circumscribed. At a graduate or research level treatment it may be necessary and even desirable for the course to veer off asymptotically into more and more difficult and obscure material, eventually reaching the 'cutting edge' of the subject (the edge where the theory no longer cuts). For an undergraduate course this will not do—and this will mean some material has to be omitted; but that is not a serious worry: what *is* required (and this goes especially for the presentation in lectures) is that subtle blend of seriousness and stimulation that cannot really be prescribed or explained, but is as rare as it is easily recognized. Whether we have fulfilled *this* requirement very satisfactorily is doubtful; but we *have* succeeded in omitting some material.

Another function that an undergraduate course must satisfy is that is should be *examinable.* This means slightly less emphasis on the sort of lengthy calculations and verifications that are typically put forth as problems in the 'trade' books (though such problems are in the right context useful and important) and more emphasis on the slightly shorter type of problem that requires some *thinking* for its solution—problems that, as G.H. Hardy might have said, show a bit of *spin.* We cannot claim that our problems are being bowled so artfully, or even that that's always what's intended;

but it *can* be pointed out that a number of the problems appearing at the ends of chapters *have* been set on past papers of undergraduate examinations at Oxford, and that these, and other problems set in the same spirit, may be useful not only for the lofty purpose of enriching one's comprehension of a noble subject, but also for the mundane but very important matter of proving to the rest of the world that one's comprehension has indeed been enriched!

A number of our colleagues have helped us in various ways in the preparation of this material—either by providing us with problems or ideas for problems, or by reading portions of the original lecture notes on which the course is based and offering criticism and useful feedback, or pointing out errors; and we would particularly like to thank David Bernstein, Tom Hurd, Lionel Mason, Tristan Needham, David Samuel, and Nick Woodhouse for this. Roger Penrose has in his publications and lectures suggested a number of points of approach and presentation that we have used or adapted, for which we offer here summarily our acknowledgements and thanks—for indeed much of the subject as it presently stands bears the imprint of his significant influence. And for the mathematical typesetting we would like to thank Jian Peng of Oxford University Computing Laboratory.

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1 Introduction

1.1 Space, time, and gravitation

GENERAL RELATIVITY is Einstein's theory of gravitation. It is not only a theory of gravity: it is a theory of the structure of space and time, and hence a theory of the dynamics of the universe in its entirety. The theory is a vast edifice of pure geometry, indisputably elegant, and of great mathematical interest.

When general relativity emerged in its definitive form in November 1915, and became more widely known the following year with the publication of Einstein's famous exposé *Die Grundlage der allgemeinen Relativitätstheorie* in *Annalen der Physik*, the notions it propounded constituted a unique, revolutionary contribution to the progress of science. The story of its rapid, dramatic confirmation by the bending-of-light measurements associated with the eclipse of 1919 is thrilling part of the scientific history. The theory was quickly accepted as physically correct—but at the same time acquired a reputation for formidable mathematical complexity. So much so that it is said that when an American newspaper reporter asked Sir Arthur Eddington (the celebrated astronomer who had led the successful solar eclipse expedition) whether it was true that only three people in the world really understood general relativity, Eddington swiftly replied, "Ah, yes—but who's the third?"

The revolutionary character of Einstein's gravitational theory lies in the change of attitude towards space and time that it demands from us. Following Einstein's extraordinary 1905 paper on special relativity (*Zur Elektrodynamik bewegter Körper*, in *Annalen der Physik*) a major step forward was taken by the mathematician Hermann Minkowski (1864-1909) who recognized that the correct way to view special relativity, and in particular the Lorentz transformation, was in terms of a single entity *space-time*, rather than a mere jumbling up of space and time coordinates.

His famous 1908 address *Space and Time* opens theatrically with the following words: "The views of space and time that I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." And how right he was. It was in Minkowski's work that many of the geometrical ideas so important to a correct, thorough understanding of relativity were first introduced—particle world-lines, space-like and time-like vectors, the forward and backward null cones—ideas that all carry over also into general relativity. Minkowski space (as the flat Lorentzian space-time manifold of special relativity is now called) was seen to be the proper arena for the description of special relativistic physical phenonena—a point of view that Einstein himself was quickly to embrace. "We are compelled to admit," writes Minkowski in the same 1908 address, "that it is only in four dimensions that the relations here taken under consideration reveal their inner being in full simplicity, and that on a three dimensional space forced upon us *a priori* they cast a very complicated projection."

1.2 The dynamics of the universe in its entirety

But general relativity goes much further, and incorporates the gravitational field into the structure of space-time itself. Since gravitational fields can vary from place to place, this means that space-time also must vary in some way from place to place. The mathematical framework that deals with geometries that vary from point to point is called *differential geometry*; and it is a particular species of differential geometry called *Riemannian geometry*—named after Bernhard Riemann (1826-1866) who among his many mathematical achievements founded the general theory of higher dimensional curved spaces—that offers the analytical basis for a description of the gravitational field. Einstein himself was to remark (in *The Meaning of Relativity*) that "the mathematical knowledge that has made it possible to establish the general theory of relativity we owe to the geometrical investigations of Gauss and Riemann."

And thus we are left to marvel that Einstein was led to such a refined, abstract branch of pure geometry for his relativistic theory of gravitation. " It is my conviction," he writes in his 1933 Herbert Spencer lecture at Oxford, "that pure mathematical construction enables us to discover the concepts, and the laws connecting them, that give us the key to the understanding of the phenomena of Nature." It was easy, perhaps, for Einstein to say this in 1933. By that time he was recognized throughout the world as a genius. His theories had transformed the shape of physical science. And yet another accolade has been bestowed upon him in the years immediately preceding—the American astronomer Edwin Hubble (1889-1953) had announced in 1929 his discovery that the universe was expanding!—that remote galaxies showed a red shift systematically correlated with their distance. This observation was very much in accord with the pattern of results suggested by general relativity, and opened the door to yet another new branch of physics: relativistic cosmology.

1.3 What is so special about general relativity?

The road to special relativity had been swift and straight, with most of the essentials accomplished in Einstein's first article on the subject, his 1905 paper. The famous $E = mc^2$ formula follows shortly thereafter in a brief note entitled *Ist die Trägheit* *eines Körpers von seinem Energiegehalt abhängig*? (Does the inertia of a body depend upon its energy-content?) which ends with the speculation that "It is not impossible that with bodies whose energy-content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test." Capped with Minkowski's mathematics, the theory was then set for a successful launch onto the high seas of twentieth century physics—and is still very much afloat.

So much for Albert Einstein (1879-1955) at age twenty-six: the genesis of general relativity, however, was a far less straightforward matter, and took the better part of a decade. One is reminded of the way in which some musical pieces seem to have sprung, as it were, fully composed from the musician's head—one gets this impression, for example, in many of the writings of Bach; whereas other pieces are clearly arrived at only after extensive revisions, with ideas being gradually assembled in the course of a tortured creative process spread over a period of some time—in this case Beethoven comes to mind as possibly the best example, and one also thinks particularly of Mahler. If special relativity belongs to the first category of composition, then general relativity certainly falls into the second.

Nevertheless, despite its complex origins, a Beethoven symphony or quartet does have a certain finality to it—a certain undeniable perfection; and much the same can be said of Einstein's gravitational theory. It has that rare quality about it that excites all of one's attentions in a physical theory: it has an air of permanence.

And it is, perhaps, this aspect of Einstein's theory that makes it (quite apart from its necessary interest to professional physicists, as a key component to our present understanding of nature) a subject worthy of intellectual enquiry by students who, after coming to understand it, will not in any ordinary sense have any practical use for it. It is a work of art.

1.4 The mercurial matter of Mercury

In December 1907 Einstein wrote to his friend Conrad Habicht (1876-1958) that he was "...busy working on relativity theory in connection with the law of gravitation, with which I hope to account for the still unexplained secular changes in the perihelion motion of the planet Mercury—so far it doesn't seem to work." Einstein, Habicht, and another friend Maurice Solovine (1875-1958) had known one another in Bern (where Einstein had taken up his job at the patent office in 1902) and had met regularly under the auspices of the *Olympian Academy* (founded by and comprising just the three of them) to discuss and debate philosophical, scientific, and literary matters. They read together from the works of Plato, Sophocles, Cervantes, Hume, Spinoza, Racine, Dickens, Mach, and Poincaré, amongst other authors. How exciting those evenings must have been! What else might they have read? (One is reminded somehow of Oscar Wilde's remark, "I have made an important discovery—that alcohol, taken in sufficient quantities, produces all the effects of intoxication.") Einstein evidently felt at ease with Habicht to discuss his ambitions and frustrations. And it was eight years later in 1915 that Einstein was able in a letter to the physicist Paul Ehrenfest (1880-1933) to report that "for a few days, I was beside myself with joyous excitement" over the correct explanation of Mercury's orbit, which he had recently obtained.

1.5 An idée fixe

But why this apparent obsession with the misbehaviour of Mercury's orbit? Why dwell on this little detail? The problem with Mercury had been known since the middle of the nineteenth century. According to Newton's theory, and as first hypothesized by Kepler, the orbit of an ideal planet is a perfect ellipse, with the Sun located at one of the foci, as illustrated below.

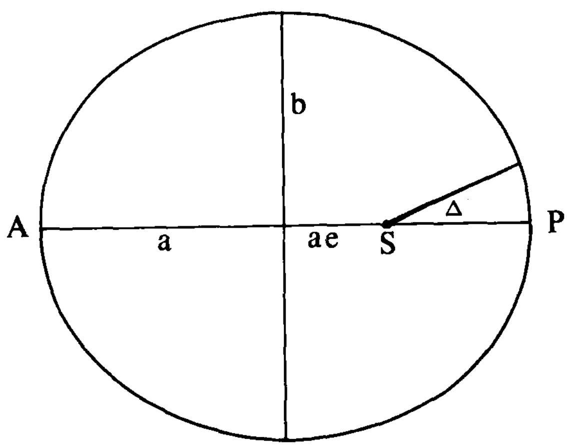

Figure 1.1. The shift of Mercury's perihelion. With each revolution the axis of the ellipse moves through a small angle Δ in the direction of revolution.

The semi-major axis a and the semi-minor axis b are related by $b^2 = a^2(1 - e^2)$, where e is the eccentricity. The sun is offset from the center of the ellipse by a distance ae, and the equation of the orbit is given by

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

where r is the distance between the sun and the planet, and θ is the angle between the line joining the sun and the planet, and the semi-major axis through S. Clearly where $\theta = 0$ the planet is at its *perihelion* (point of closest approach) with $r = a(1 - e)$; whereas when $\theta = \pi$ the planet is most distant, at its aphelion, with $r = a(1 + e)$.

In reality planetary orbits are not perfect ellipses, owing primarily to the perturbing effects of other planets. This was of course well-appreciated by Newton, who in *De Motu* (version IIIB) observes that "...the planets neither move exactly in an ellipse, nor revolve twice in the same orbit—there are as many orbits to a planet as it has revolutions—and the orbit of any one planet depends on the combined motion of all of the planets, not to mention the action of all these on each other. But to consider simultaneously all the causes of motion and to define these motions by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the force of the entire human intellect."

Fortunately, the perturbing effects are relatively small, and one can treat the orbits as approximately elliptical, studying the deviation from perfect ellipticity, as induced by effects such as those mentioned by Newton. Notable among these is the effect of a small rotation in the axis of the ellipse, which can be measured by the angle Δ by which the perihelion shifts, per revolution, from its previous position. For Mercury, planetary influences result in a perihelion shift of roughly 500″ (seconds of arc) per century. Since Mercury's period is about one quarter that of the Earth, this works out to about 1.25″ per orbit—not very much! Around 260,000 years are required for the precession to go all the way around. But it does happen.

The anomaly in Mercury's orbit was discovered by the French astronomer Urbain Jean Joseph Le Verrier (1811-1877) in 1859. He showed that there was a discrepancy between observation and theory—by a figure which (according to present-day measurements) amounts to an excess motion in the perihelion shift of about 43″ per century. This was the 'still unexplained change' in Mercury's orbit that had caught Einstein's attention in 1907.

1.6 Beside himself with joy

The history of attempts to explain this phenomenon is an elaborate affair, and makes a very interesting chapter in the history of astronomy. In essence, either a mysterious new planet, or some form of hidden quasi-planetary material, had to be present—or the laws of gravity needed to be modified.

On the latter point it is indeed straightforward enough to 'induce' a systematic perihelion shift by means of a slight modification of Newton's laws: Newton himself had noted, for example, that if the gravitational force obeyed not an inverse square law but rather, say, a modified force-law of the form

$$F = \frac{\alpha r^m - \beta r^n}{r^3}$$

where α, β, m, n are constants, then the angle θ between successive perihelia ($\theta = 2\pi + \Delta$) is given by

$$\theta = 2\pi \left(\frac{\alpha - \beta}{m\alpha - n\beta} \right)^{1/2}$$

Thus in particular if $F = \alpha r^{m-3}$ we have $\theta = 2\pi m^{-1/2}$. By today's way of thinking (which as a consequence of Einstein's scientific work has become much more Platonic) such a modification of Newton's theory strikes us as rather vulgar—but a hundred years ago the approach was taken seriously as a possible explanation of Mercury's anomaly. It doesn't work.

Einstein, however, was able in his theory to deduce a very elegant formula for the perihelion effect, given by

$$\Delta = \frac{24\pi^3 a^2}{T^2 c^2 (1 - e^2)}$$

where T is the period of the orbit, and c is the speed of light. No wonder, given the accuracy with which it accounts for the observed orbits, that Einstein was beside himself with joy at the discovery of this relation. One can imagine the profound shock it must have given him to have encountered such a vivid confirmation of his ideas—a confirmation of the sublime relations holding between the abstractions of the space-time continuum, and something so down to earth as the science of the solar system.

1.7 Rudis indigestaque moles

This may give us an intimation as to why the theory has been lifted to such preeminent esteem by the cognoscente of successive generations. By comparison, the scope of other physical theories, indeed much of science as a whole, takes on the character of 'a rough and confused mass.'

General relativity is a theory of some complexity, and it does involve a good deal of fairly difficult mathematics. Nevertheless it is possible—providing one is willing to take a number of details on faith—to present an overview of the theory, reducing it to its most basic mathematical elements.

Space-time, according to Einstein's theory, is a *four-dimensional manifold* (the 'space-time continuum'). The manifold looks locally like a piece of R^4, but there are two important distinctions: the various 'pieces' do not necessarily fit together to form a *global* R^4 (what they do form is typically something more complicated); and even locally the geometry is not Euclidean, nor even flat (like the Lorentzian geometry of special relativity).

The space-time is covered by a series of coordinate patches U_i, and in each coordinate patch we have a set of coordinates x^a $(a = 0, 1, 2, 3)$. The basic, underlying geometry of the manifold (its *differentiable structure*) is determined by the relations holding between systems of coordinates in overlapping patches.

The mathematical tool used for studying manifolds is called *tensor calculus*. A tensor is a sort of a many-index analogue of a vector. Differentiation of tensors is a intricate matter since the value of the derivative of a tensor can apparently depend (in a coordinate overlap region) on which set of coordinates is used to perform the differentiation. This situation is remedied by the introduction in each coordinate patch of a special three-index array of functions denoted Γ^a_{bc} called the 'connection'. The connection is required to transform in a particular way in coordinate transition (i.e. overlap) regions. The correct derivative of a tensor is then taken by means of a slightly complicated operation that involves systematic use of these special connection symbols. The resulting process—called 'covariant differentiation' has a multitude of natural, compelling features. (The covariant derivative of a tensor A^{bc} is denoted $\nabla_a A^{bc}$ to distinguish it from the array of partial derivatives $\partial_a A^{bc}$ where $\partial_a = \partial/\partial x^a$. More explicitly, the covariant derivative $\nabla_a A^{bc}$ is given by an expression of the form $\partial_a A^{bc} + \Gamma^b_{ap} A^{pc} + \Gamma^c_{aq} A^{bq}$.) When the operations of calculus in this way become well-defined we say that the space-time has the structure of a *differentiable manifold with connection.*

1.8 The metric tensor

But in what sense is the manifold a *space-time* manifold? In special relativity the metrical properties of space and time are determined by a 'Lorentzian' metric with a pseudo-Euclidean signature, given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

In the case of a material particle the infinitesimal interval ds represents the change in 'proper time' (i.e. time as measured by its own natural clock) experienced by the particle when in undergoes a displacement in space-time given by (dt, dx, dy, dz). Units are chosen such that the speed of light is one. Note that if dx, dy, and dz vanish, then the proper time s agrees with the coordinate t. Thus t can be interpreted as the time measured by an observer at rest at the origin in this system of coordinates. But if dx is greater than zero, say, then ds must be less than dt. So we see that while an observer at the origin measures an interval of time dt, a moving body instead measures the interval given by $ds^2 = dt^2 - dx^2$, or more explicitly $ds = (1-v^2)^{\frac{1}{2}} dt$ where $v = dx/dt$ is the velocity of the moving observer relative to the origin. And thus time seems to be going more 'slowly' for the moving particle.

The infinitesimal interval of special relativity can be written more compactly by use of an index notation in the form

$$ds^2 = \eta_{ab} dx^a dx^b$$

where dx^a $(a = 0, 1, 2, 3)$ is the space-time displacement, and η_{ab} is the flat metric of special relativity with diagonal components $(1, -1, -1, -1)$. Summation is implied over the repeated indices. In general relativity the idea is that the geometry of space-time varies from point to point—and this is represented by allowing the metric to be described by a tensor field g_{ab} that varies over the space-time. The infinitesimal interval is then given by

$$ds^2 = g_{ab} dx^a dx^b,$$

where g_{ab} is a four-by-four symmetric, non-degenerate matrix. It was Einstein's key recognition that the gravitational field could be embodied in the specification of the space-time manifold M and its 'curved' Lorentzian metric g_{ab}. The idea that M is a *space-time* is implicit in the requirement that g_{ab} should have signature $(+, -, -, -)$; i.e. that it should have three negative eigenvalues, and one positive eigenvalue.

One immediate consequence of the formula $ds^2 = g_{ab}dx^a dx^b$ is that the proper time experienced by a particle depends on the nature of the gravitational field through which it may be passing. This leads to a gravitational *time dilation* effect, which is one of the important features of the theory—when light is emitted in the neighbourhood of a strong gravitational field (e.g. near the Sun) it is seen to be *red-shifted* when received in the vicinity of a weaker field (e.g. at the Earth's surface).

1.9 The Levi-Civita connection

The flat space-time of special relativity is called *Minkowski space,* and g_{ab} can be regarded as 'fixed' throughout the manifold. But in a curved space-time g_{ab} varies from point to point in an *essential* way.

Therefore to set up a workable physical theory one needs a relation between the metrical properties of the space-time (as determined by g_{ab}) and the operations of tensor calculus (as determined by the connection Γ^c_{ab}). This is established by a powerful result known as the *fundamental theorem of Riemannian geometry.* According to this theorem the space-time metric determines the associated connection Γ^c_{ab} according to a remarkable formula, given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

where g^{ab} is the inverse of g_{ab} (so $g^{ab}g_{bc} = \delta^a_c$), and ∂_a again denotes $\partial/\partial x^a$. The connection thus determined is called the *Levi-Civita* connection, and the corresponding covariant derivative ∇_a has the important property that when applied to the metric tensor it gives the result zero: $\nabla_a g_{bc} = 0$, i.e. the metric is 'covariantly constant'. The point of the theorem is that given g_{ab} the connection is determined uniquely by this property.

1.10 The field equations

Sitting at the apex of the theory are Einstein's equations for the gravitational field. These are the equations that relate g_{ab} to the local distribution of matter, and are thus in many respects analogous to the Newton-Poisson equation $\nabla^2\Phi = 4\pi G\rho$ which relates the grativational potential Φ to the matter density ρ, where G is the gravitational constant. But is should be stressed that Einstein's equations amount to rather more than a mere 'relativistic upgrade' of the Newtonian equation—this will become apparent as details of the theory become understood.

The essence of Einstein's theory can be understood as follows. According to the classical theory of continuum mechanics, the equations of motion and the conservation laws for energy, momentum, and angular momentum are embodied in the requirement that a special tensor T^{ab} called the *stress tensor* should be 'conserved'—conserved in the sense that its divergence $\nabla_a T^{ab}$ vanish. Now to grasp this requires something of a leap in both faith and imagination, since the idea applies to a variety of physical theories, and when these theories are cast in their relativistic form the specification of T^{ab} is not always an obviously well-posed problem with a unique answer. Nevertheless for practical purposes one *can* account for the matter content of space-time by the specification of a symmetric tensor T^{ab} with vanishing divergence. For example, in the case of an ideal fluid we have

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab}$$

where ρ is the energy density, p is the pressure, and u^a is the four-velocity field. The vanishing of the divergence of T^{ab} leads to Euler's equations of motion for the fluid, and to the conservation equations for the energy of the fluid.

Now it turns out that from Γ^a_{bc} one can build up a special tensor G^{ab} that automatically has vanishing divergence. This tensor is called the *Einstein tensor*, and is given explicitly by a simple formula involving terms linear in the first derivatives of Γ^a_{bc} and terms quadratic in Γ^a_{bc} itself. Einstein was led in his investigations to propose that G^{ab}, which is built up geometrically from g_{ab}, must be *proportional* to the stress tensor, and that the factor of the proportionality should be determined by the gravitational constant:

$$G^{ab} = 8\pi G T^{ab}.$$

In this way T^{ab} acts as the 'source' of the gravitational field (as does ρ in the Newton-Poisson equation), whereas the metric g_{ab} thereby determined acts itself on the matter distribution through the requirement that the divergence $\nabla_a T^{ab}$ vanishes where ∇_a is the Levi-Civita connection determined by g_{ab}. And at the same time the matter distribution T^{ab} can depend algebraically on properties of g_{ab}, as seen for example in the case of the fluid stress tensor illustrated above.

Thus Einstein's equations are riddled with non-linearities. This has a number of consequences—not least of which are the difficulties encountered in the construction of exact solutions. And Newton's remarks on the complexities of the many-body problem apply in general relativity even to the two-body problem—since a 'third body' does in effect appear in the form of *gravitational radiation*! Nevertheless many exact solutions *are* known, and much is known now even about situations where it has not been possible to arrive at a complete description. It is worth bearing in mind that although Einstein's theory remains unchanged in its basic content since its origination in 1915, nevertheless a good deal of work has gone on in the meanwhile, and much is understood now that previously lay shrouded in obscurity, or was simply unknown on account of the lack of appropriate mathematical tools.

But more cannot be said without some systematic development of these tools—a task to which we now turn.

2 Vectors and tensors in flat three-space: old wine in a new bottle

'I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index that occurs twice ...'

—Albert Einstein (remark made to a friend)

2.1 Cartesian tensors: an invitation to indices

LOCAL DIFFERENTIAL GEOMETRY consists in the first instance of an amplification and refinement of tensorial methods. In particular, the use of an *index notation* is the key to a great conceptual and geometrical simplification. We begin therefore with a transcription of elementary vector algebra in three dimensions. The ideas will be familiar but the notation new. It will be seen how the index notation gives one insight into the character of relations that otherwise might seem obscure, and at the same time provides a powerful computational tool.

The standard Cartesian coordinates of 3-dimensional space with respect to a fixed origin will be denoted x_i $(i = 1,2,3)$ and we shall write $\mathbf{A} = A_i$ to indicate that the components of a vector $\mathbf{A}$ with respect to this coordinate system are A_i. The magnitude of $\mathbf{A}$ is given by $\mathbf{A} \cdot \mathbf{A} = A_i A_i$. Here we use the *Einstein summation convention*, whereby in a given term of an expression if an index appears twice an automatic summation is performed: no index may appear more than twice in a given term, and any 'free' (i.e. non-repeated) index is understood to run over the whole range. Thus $A_i A_i$ is an abbreviation for $\sum_i A_i A_i$, and the scalar product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A} \cdot \mathbf{B} = A_i B_i$.

Multiple index quantities often arise out of problems in geometry and physics. The most basic of these is the *Kronecker delta* δ_{ij} defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It is essentially the identity matrix, and as a consequence can readily be seen to satisfy $\delta_{ij} = \delta_{ji}$, $\delta_{ij}\delta_{jk} = \delta_{ik}$, $\delta_{ii} = 3$, and $\delta_{ij} A_j = A_i$ for any vector A_i.

Another important multiple index quantity is the permutation tensor or *epsilon tensor*, defined by:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123} \\ \text{-1} & \text{if } ijk \text{ is an odd permutation of 123} \\ 0 & \text{otherwise, i.e. if } ijk \text{ are not all distinct.} \end{cases}$$

One readily verifies that $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ and that $\varepsilon_{ijk} = -\varepsilon_{jik}$, and $\varepsilon_{iij} = 0$.

Most of the basic identities of vector algebra and vector calculus arise as a consequence of a special relation that holds between δ_{ij} and ε_{ijk}, called the *contracted epsilon identity*:

$$\varepsilon_{iab}\varepsilon_{ipq} = \delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}. \tag{2.1.1}$$

Thus when two epsilon tensors are 'contracted' together over their first indices the result can be decomposed into a sum of expressions involving the Kronecker delta. The result is sufficiently basic that it is worth memorizing. Examples of its utility follow forthwith.

The vector product or wedge product $\mathbf{C} = \mathbf{A} \wedge \mathbf{B}$ of two vectors can be expressed by use of epsilon as follows:

$$C_i = \varepsilon_{ijk}A_jB_k. \tag{2.1.2}$$

The scalar triple product of three vectors is

$$\varepsilon_{ijk}P_iQ_jR_k = \mathbf{P}\cdot(\mathbf{Q}\wedge\mathbf{R}) = [\mathbf{P},\mathbf{Q},\mathbf{R}]. \tag{2.1.3}$$

Note how the cyclic property of the scalar triple product

$$\mathbf{P}\cdot(\mathbf{Q}\wedge\mathbf{R}) = \mathbf{Q}\cdot(\mathbf{R}\wedge\mathbf{P}) = \mathbf{R}\cdot(\mathbf{P}\wedge\mathbf{Q})$$

follows at once from the expression $\varepsilon_{ijk}P_iQ_jR_k$ by virtue of the identity

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}.$$

In the case of the repeated vector product $\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})$ we derive the following familiar identity:

$$\begin{aligned}\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) &= \varepsilon_{ijk}A_j(\varepsilon_{kpq}B_pC_q)\\ &= \varepsilon_{kij}\varepsilon_{kpq}A_jB_pC_q\\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})A_jB_pC_q\\ &= B_iA_qC_q - C_iA_pB_p\\ &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).\end{aligned} \tag{2.1.4}$$

Note how simply this identity follows from the contracted epsilon identity. In fact, the argument is reversible, and the known validity of (2.1.4) establishes a proof of (2.1.1). Alternatively, (2.1.1) may be established directly by evaluation component by component.

Further identities may be readily constructed by use of the contracted epsilon identity. For example:

$$\begin{aligned}(\mathbf{A}\wedge\mathbf{B})\cdot(\mathbf{C}\wedge\mathbf{D}) &= (\varepsilon_{ijk}A_jB_k)(\varepsilon_{ipq}C_pD_q)\\ &= \varepsilon_{ijk}\varepsilon_{ipq}A_jB_kC_pD_q\\ &= (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})A_jB_kC_pD_q\\ &= A_pC_pB_qD_q - A_qD_qB_pC_p\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).\end{aligned} \tag{2.1.5}$$