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INTRODUCTION TO MATHEMATICAL ANALYSIS

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PREFACE

This book has been written to serve as a text for students in a first course in mathematical analysis. Such a course would usually follow rather quickly the traditional freshman calculus and generally appears in college catalogs under the name of Introduction to Analysis, Mathematical Analysis, or Advanced Calculus. The body of material which has come to be known as freshman calculus serves as a prerequisite. This presumes that the reader is equipped with good skills in advanced high school or college algebra and in trigonometry.

Today calculus serves a much more diverse audience than in years past. Not only are mathematics and natural science majors and engineering students taking calculus but there are growing numbers of business, statistics, and computer science students in these courses. Traditionally, the purpose of freshman calculus has been to teach the student the facts and applications of calculus. This means that the student acquires skill in the mechanics of calculus and a certain level of proficiency at using calculus in those many areas where the methods of calculus prove useful. Often, however, little time in the freshman calculus course is spent on the theory which enhances one's understanding of the subject. This is especially true today, when the trend has been to expose the student as early as possible to a widening variety of calculus applications. It is the role of the sophomore or junior level mathematical analysis or advanced calculus course to provide the understanding that is so often lacking when the student leaves the freshman calculus course.

The authors believe that the purpose of advanced calculus is twofold: (1) to allow the student to become acquainted with, and develop a certain level of proficiency in, the techniques and methods of mathematical analysis (sometimes the proof is more important than the theorem) and (2) to be able to use these techniques and methods to reinforce and solidify an understanding of the learned calculus results. There is not an abundance of new material or facts to be learned; the emphasis instead is on looking at much the same calculus topics in greater depth and with a definite direction toward understanding. At the end of such a course the student should not only be better at doing and using calculus but should be well versed in the methods and techniques of mathematical analysis. This, after all, is the substance that provides an appreciation for the theory of calculus and lays the foundation for more advanced work in the mathematical sciences.

Since this book is intended for the reader who has not previously seen a theoretical and rigorous development of calculus, we have included numerous illustrative examples with detailed explanations. In the initial chapters some of the proofs tend to be expository in style, with a necessary sacrifice of elegance. This is done to provide more insight into the construction of mathematical proofs and to help develop the skills used in proving statements in mathematics. Many of the exercises call upon the student's ability to use the methods and techniques employed in the text, and, conversely, working the exercises provides a deeper and more thorough understanding of the theorems and their proofs. The exercises should be dealt with deliberately as they serve as an integral part of the text.

We have found that the first seven chapters provide a good arrangement of topics for a one-semester course in mathematical analysis. Chapter 1 contains a development of the real number system. If a developmental approach is desired then all the sections should be covered. For an axiomatic approach sections 1.2 to 1.4 can be skipped without loss of continuity. In either case the reader should strive for an overview of this material rather than a detailed exposition of each and every step in the construction process. Chapter 2 develops sequences and sets as tools to use in the calculus. The core of the material is in Chaps. 3 to 7, which should be covered in detail, except perhaps the optional sections (4.4, 5.4, and 6.4).

In a two-semester course the instructor may wish to include some or all of the optional sections, together with the last three chapters in the text. We have found that this can adequately be covered at a leisurely pace in such a one-year course. If it is desired to place more emphasis on multivariate calculus, Chaps. 8 and 9 should be supplemented with topics of the instructor's choosing, and Chap. 10 may become optional, depending on the audience and the direction of the particular course.

We would like to thank Professor George Springer for his many helpful suggestions during the various stages of development of the manuscript. We also appreciate the efforts of our typists, Elizabeth Bator, Betty Leszczak, and Amy Raskin.

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Philip W. Zipse

CONTENTS

* Optional sections.

CHAPTER

ONE

REAL NUMBERS AND FUNCTIONS

1.1 SETS AND FUNCTIONS

In the study of analysis, as in many other areas of mathematics, the concepts of *set* and *function* are fundamental. Every language has certain words which are basic and remain undefined but of whose meaning there is universal acceptance. In mathematics the word "set" is such a term; a set is understood to be a *well-defined* collection of objects called *elements*. The term well-defined just means for us that some mechanism exists whereby one is able to determine whether or not a given element belongs to the set.

We denote sets by capital letters A, B, C, etc., and use lowercase letters a, b, c, etc., to represent elements. If an element x belongs to or (equivalently) is a member of the set S, we write $x \in S$; to designate that the element x does not belong to S we write $x \notin S$.

If each element in the set A is also a member of the set B, we say that A is a subset of B and write $A \subseteq B$ or equivalently $B \supseteq A$. We call two sets A and B *equal* and write $A = B$ provided $A \subseteq B$ and $B \subseteq A$. Two sets are equal, then, if the sets consist of precisely the same elements. If $A \subseteq B$ and $A \neq B$, we say that A is a proper subset of B and designate this by $A \subset B$ (equivalently $B \supset A$). If $A \subset B$, then every element in A is also in B but there is at least one element in B which fails to be in A.

In any discussion involving sets the letter U denotes the *universal set*, which is the set of all elements under discussion; the symbol $\varnothing$ denotes the *empty set*, that set which contains no elements. Then for every set A we have

$$\varnothing \subseteq A \subseteq U$$

Specific sets can be defined by listing all the elements in the set or by stating a characteristic property which is unique to those elements belonging to the set. For example

$$G = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$$

can be alternately defined by

$$G = \{x \mid x \text{ is one of the first five letters of the Greek alphabet}\}$$

which is read "G is the set of all elements x such that x is one of the first five letters of the Greek alphabet." When elements in a set are listed, each element should be listed exactly once; order is not important.

We use the above notation to define some natural ways of combining sets to construct new sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$$

$A \cup B$ is called the *union* of A and B, read "A union B."

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$A \cap B$ is called the *intersection* of A and B, read "A intersection B."

$$A' = \{x \mid x \in U \text{ and } x \notin A\}$$

A' is called the *complement* of A (sometimes denoted $U - A$).

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

$A \times B$ is called the *cartesian product* of the sets A and B.

We can also consider the union and intersection of more than two sets. The union of a class of sets is just that set consisting of all those elements which belong to at least one set in the class. The intersection of a class of sets is the set consisting of all those elements which belong to each and every set in the class.

The cartesian product of A and B is the set of all ordered pairs (a, b), where a is any element in A and b is any element in B. Recall that the concept of ordered pairs of numbers was needed when graphing equations in the (x, y) plane. The plane is a geometric model of a cartesian product, and the graph of an equation is a subset of this cartesian product. The geometry allows us to visualize graphs and enhances our understanding of important concepts in algebra and calculus.

Definition 1.1 A *function* is a nonempty set X, a nonempty set Y, and a rule of correspondence f which associates with each element $x \in X$ a unique element $y \in Y$.

The element y associated with a given element $x \in X$ is denoted $f(x)$, and we write $y = f(x)$; y is called the *image of* x *under* f, and x is called a *preimage of* y. The function is often denoted by $f : X \to Y$ or sometimes just by f when the sets X, Y are clear from the context. The set X is called the *domain* of the function and the set $f(X) \subseteq Y$, defined by

$$f(X) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

is called the *range* of the function. If $f(X) = Y$, we say that the function is *onto* Y. The graph of the function, written $gr(f)$, is a subset of the cartesian product $X \times Y$ and is defined by

$$gr(f) = \{(x, f(x)) \mid x \in X\}$$

As mentioned earlier, if X and Y are sets of numbers, the graph of f can be (and often is) visualized as a subset of the (x, y) plane. Some authors find it convenient to identify a function with its graph and to think of a function as a special kind of subset of the cartesian product $X \times Y$.

A function $f : X \to Y$ is called *one-to-one* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, that is, if no two distinct elements in the domain of f are assigned to the same element in the range. Thus each range element has a unique preimage. One-to-one functions are useful because they allow us to define a new function called the *inverse function*. If $f : X \to Y$ is one-to-one, we define the function $f^{-1} : f(X) \to X$, read "f inverse," by the following: For each $y \in f(X)$ define $f^{-1}(y)$ to be the unique preimage of y under f. Then $f^{-1}(y) = x$ if and only if $f(x) = y$. It is clear that f^{-1} is onto X and that f^{-1} is one-to-one. If $f : X \to Y$ is one-to-one and onto Y then f is called a *one-to-one correspondence* between X and Y. In this case $f^{-1} : Y \to X$ is a one-to-one correspondence between Y and X.

Just as we can combine sets to construct new sets, we can combine given functions in the following way: If $f : X \to Y$ and $g : Y \to Z$, we define the *composite function* $g \circ f : X \to Z$ by $g \circ f(x) = g(f(x))$ for every $x \in X$. Thus composition of two functions is defined as the sequential action of the individual functions. For example, let f be the function which converts temperature in degrees Fahrenheit to temperature in degrees Celsius and let g be the function which converts temperature in degrees Celsius into absolute temperature in kelvins. Then $g \circ f$ is the function which converts temperature in degrees Fahrenheit into temperature in kelvins.

Two functions $f : X \to Y$ and $g : X \to Y$ are said to be *equal* and we write $f = g$ provided $f(x) = g(x)$ for every $x \in X$. If $f : X \to Y$, we shall sometimes want to consider f restricted to some nonempty subset $A \subseteq X$. We define $f_A : A \to Y$ by $f_A(x) = f(x)$ for every $x \in A$. If f is one-to-one then so is f_A for each nonempty $A \subseteq X$, and if f_A is onto Y for some nonempty $A \subseteq X$ then so is f onto Y. The function $i_X : X \to X$ defined by $i_X(x) = x$ for every $x \in X$ is called the *identity function on* X. i_X "maps" each element $x \in X$ onto itself. It is clear that i_X is one-to-one and onto X and that $i_X^{-1} = i_X$. If $f: X \to Y$ is one-to-one then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_{f(X)}$. In particular, if $f: X \to X$ is one-to-one and onto X then we have that $f^{-1} \circ f = f \circ f^{-1} = i_X$.

If X is a nonempty set, a *relation* R in X is a nonempty subset of $X \times X$; that is, $\varnothing \neq R \subseteq X \times X$. We have already encountered some relations in X; if $f: X \to X$ then $\mathrm{gr}(f)$ is a relation in X. Of course, not every relation in X is the graph of some function. In order for the relation R in X to be the graph of a function it is necessary (and sufficient) that for each $x \in X$, R contains exactly one ordered pair with first component x. Graphs of functions are important relations, but there are other kinds of relations that are equally important. We investigate one type of relation, called an equivalence relation, in the following paragraph.

If R is a relation in X, we write $x \sim y$ provided $(x, y) \in R$.

Definition 1.2 A relation R in X is called an *equivalence relation* if:
(*a*) $x \sim x$ for every $x \in X$ (reflexive property).
(*b*) $x \sim y$ implies $y \sim x$ for all $x, y \in X$ (symmetric property).
(*c*) $x \sim y$ and $y \sim z$ implies $x \sim z$ for all $x, y, z \in X$ (transitive property).

Suppose $\mathcal{F} = \{f \mid f: X \to Y\}$; that is, $\mathcal{F}$ is the set of all functions f with domain X and range contained in Y. Let $x_0 \in X$ be fixed. We define the following relation R in $\mathcal{F}$: $(f, g) \in R$ (equivalently, $f \sim g$) provided $f(x_0) = g(x_0)$. It is clear that $f \sim f$ for every $f \in \mathcal{F}$, and if $f \sim g$ then $f(x_0) = g(x_0)$; hence $g(x_0) = f(x_0)$, and so $g \sim f$. Moreover, if $f \sim g$ and $g \sim h$ then $f(x_0) = g(x_0)$ and $g(x_0) = h(x_0)$. Therefore $f(x_0) = h(x_0)$ and $f \sim h$. It follows that R is an equivalence relation in $\mathcal{F}$.

A *partition* of a nonempty set X is a class of nonempty subsets of X which has the following property: Each element in X belongs to exactly one set in the class. One of the important facts about any equivalence relation in X is that it induces a partition of the set X. Suppose R is an equivalence relation in X. For each $x \in X$ we define the *equivalence class* of x by

$$[x] = \{y \in X \mid y \sim x\}$$

Theorem 1.1 The distinct equivalence classes of an equivalence relation in X form a partition of X.

Proof Let R be an equivalence relation in X. Each equivalence class $[x]$ is a subset of X (by definition) and is nonempty since $x \in [x]$ (reflexive property). Moreover, each $x \in X$ is in at least one equivalence class, namely $x \in [x]$. It remains to show that an element $x \in X$ cannot be in two distinct equivalence classes. Suppose that $x \in [y]$. Then $x \sim y$ and $y \sim x$ by the symmetry property. Now, if $z \in [y]$ then $z \sim y$, and so $z \sim x$ by the transitive property. Thus $z \in [x]$, and so $[y] \subseteq [x]$. Similarly, if $z \in [x]$ then $z \sim x$ and by the transitive property $z \sim y$. Thus $z \in [y]$, and so $[x] \subseteq [y]$. It follows that $[y] = [x]$, and so $[x]$ is the only equivalence class containing x.

In our previous example of an equivalence relation in the set $\mathcal{F}$ of all functions f with domain X and range contained in Y, each equivalence class consists of all the functions in $\mathcal{F}$ which map x_0 to the same element in Y. Theorem 1.1 has a partial converse, which says that any partition of the nonempty set X induces an equivalence relation in X for which the distinct equivalence classes are precisely the sets in the partition. To see this define

$$R = \{(x, y) \mid y \text{ belongs to the same set in the partition as } x\}$$

The reflexive, symmetric, and transitive properties are easily verified. Therefore, in this sense, there is little difference between an equivalence relation in X and a partition of X.

The next example of an equivalence relation is useful in that it allows us to gauge the size of infinite sets. We define this equivalence relation below and return to a discussion of infinite sets in Sec. 1.6.

Let U be some fixed universal set and let $\mathcal{C}$ be a class of subsets of U ($\mathcal{C}$ is a set whose elements are subsets of U). We define a relation in $\mathcal{C}$ as

$$R = \{(A, B) \mid A, B \in \mathcal{C} \text{ and there exists a one-to-one correspondence } f: A \to B\}$$

Again we write $A \sim B$ when $(A, B) \in R$. For each $A \in \mathcal{C}$, $A \sim A$ since i_A is a one-to-one correspondence between A and itself. If $A \sim B$ then there is a one-to-one correspondence $f: A \to B$. We noted earlier (also see Exercise 1.4) that f^{-1} is a one-to-one correspondence between B and A and so $B \sim A$. If $A \sim B$ and $B \sim C$ then there exist one-to-one correspondences $f: A \to B$ and $g: B \to C$. The function $g \circ f: A \to C$ is one-to-one and onto C (see Exercises 1.5 and 1.6), and so $A \sim C$. Therefore the reflexive, symmetric, and transitive properties hold, and so R is an equivalence relation in $\mathcal{C}$.

In the next three sections we construct the set $\mathbf{R}$ of all real numbers. This material can be omitted if the reader chooses, and there is no loss in continuity in going directly to Sec. 1.5, where the important properties of $\mathbf{R}$ are summarized.

EXERCISES

1.1 Show that if $f: X \to Y$ is one-to-one then $(f^{-1})^{-1} = f$.

1.2 Show that if $f: X \to Y$ is onto Y then there exists a function $g: Y \to X$ such that $f \circ g = i_Y$.

1.3 Show that if $f: X \to Y$ is one-to-one then there exists a function $h: Y \to X$ such that $h \circ f = i_X$. Verify that $h_{f(X)} = f^{-1}$.

1.4 Show that if $f: X \to Y$ is a one-to-one correspondence between X and Y then $f^{-1}: Y \to X$ is a one-to-one correspondence between Y and X.

1.5 Show that if $f: X \to Y$ is one-to-one and $g: Y \to Z$ is one-to-one then $g \circ f: X \to Z$ is one-to-one.

1.6 Show that if $f: X \to Y$ is onto Y and $g: Y \to Z$ is onto Z then $g \circ f: X \to Z$ is onto Z.

1.7 Let X be the set of all residents of New Jersey. Determine which of the following are equivalence relations in X:

(*a*) $x \sim y$ provided y has the same natural parents as x.
(*b*) $x \sim y$ provided y lives within 5 miles of x.
(*c*) $x \sim y$ provided y has the same date of birth as x.
(*d*) $x \sim y$ provided y is a brother of x.

1.8 Find all functions $f: X \to X$ such that the graph of f, $\text{gr}(f)$, is an equivalence relation in X. Describe the equivalence classes.

1.9 Describe in what sense a function $f: X \to X$ can be considered as an example of a relation in X.

1.10 Show that if X and Y are nonempty sets then $X \times Y \sim Y \times X$, where $\sim$ is the equivalence relation from the final example of Sec. 1.1.

1.11 Let X be any set and let $P(X)$, called the *power set* of X, be the set of all subsets of X. Prove that there is no one-to-one correspondence $f: X \to P(X)$.

1.2 THE NATURAL NUMBERS

We begin our development of real numbers with the set **N** of natural numbers 1, 2, 3, ... All of us have had far more than merely a casual acquaintance with this set; indeed, our initial experiences in mathematics dealt mostly with counting and the arithmetic of natural numbers. We could certainly list many properties of natural numbers, properties which we first encountered in our beginning years in elementary school or even earlier. But if we are going to outline a development of the set of all real numbers, we must be quite specific about which properties of the set **N** will be assumed to be true. From these assumptions or axioms the other familiar properties will follow.

Let **N** be a set whose elements we shall call natural numbers. We take the statements P1 to P5 as our axioms:

P1. $1 \in \mathbf{N}$; that is, **N** is a nonempty set and contains an element we designate as 1.

P2. For each element $n \in \mathbf{N}$ there is a unique element $n^* \in \mathbf{N}$ called the *successor* of n.

P3. For each element $n \in \mathbf{N}$, $n^* \neq 1$; that is, 1 is not the successor of any element in **N**.

P4. For each pair $n, m \in \mathbf{N}$ with $n \neq m$, $n^* \neq m^*$; that is, distinct elements in **N** have distinct successors.

P5. If (*a*) $A \subseteq \mathbf{N}$, (*b*) $1 \in A$, and (*c*) $p \in A$ implies $p^* \in A$ then $A = \mathbf{N}$.

These five axioms are called the *Peano postulates*, and all the known properties of natural numbers can be shown to be consequences of them. P5, called the *principle of mathematical induction*, is an important tool in many mathematical proofs. It often appears in the following form.

If for each natural number n, $S(n)$ is a statement which depends on n then in order to prove that $S(n)$ is a true statement for each and every natural number n we define the set A to be the set of all those natural numbers n for which $S(n)$ is true ($A \subseteq \mathbf{N}$). If we can show that $S(1)$ is true ($1 \in A$) and if we can show that the truth of $S(p)$ implies that $S(p^*)$ is true ($p \in A$ implies $p^* \in A$) then it follows from the principle of mathematical induction that $S(n)$ is true for every natural number n ($A = \mathbf{N}$).

The axioms allow us to name the natural numbers in the conventional way.

(*a*) $1 \in \mathbf{N}$ by P1.
(*b*) $1^* \in \mathbf{N}$ by P2 and $1^* \neq 1$ by P3. Name $1^* = 2$; then 1, 2 are distinct natural numbers.
(*c*) $2^* \in \mathbf{N}$ by P2, and $2^* \neq 1$ by P3. $2^* \neq 2$ by P4 (since $2 \neq 1$). Name $2^* = 3$; then 1, 2, 3 are distinct natural numbers.
(*d*) $3^* \in \mathbf{N}$ by P2, and $3^* \neq 1$ by P3. $3^* \neq 2$ by P4 (since $3 \neq 1$). $3^* \neq 3$ by P4 (since $3 \neq 2$). Name $3^* = 4$; then 1, 2, 3, 4 are distinct natural numbers.

By continuing to name natural numbers in this way indefinitely, we get a set $A = \{1, 2, 3, 4, \ldots\}$ of distinct natural numbers. Since A satisfies the induction hypothesis of P5, it follows from Axiom P5 that $A = \mathbf{N}$ and so every natural number has been named. Therefore

$$\mathbf{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

Next we develop an "arithmetic" in $\mathbf{N}$ by defining two binary operations called addition ($+$) and multiplication ($\cdot$).

Definition 1.3 Addition:

$$n + 1 = n^* \qquad \text{for each } n \in \mathbf{N}$$

and

$$n + p^* = (n + p)^* \qquad \text{for each } n \in \mathbf{N} \text{ and } p \in \mathbf{N}$$

Multiplication:

$$n \cdot 1 = n \qquad \text{for each } n \in \mathbf{N}$$

and

$$n \cdot p^* = (n \cdot p) + n \qquad \text{for each } n \in \mathbf{N} \text{ and } p \in \mathbf{N}$$

Notice that the operations of addition and multiplication are defined inductively: the definition of addition first gives the sum $n + 1$ and then the sums $n + 2 = (n + 1)^*$, $n + 3 = (n + 2)^*$, $n + 4 = (n + 3)^*$, etc. Similarly, the definition of multiplication first gives $n \cdot 1$ and then the products $n \cdot 2 = (n \cdot 1) + n$, $n \cdot 3 = (n \cdot 2) + n$, $n \cdot 4 = (n \cdot 3) + n$, etc. Thus, it should come as no surprise that the proofs of the various properties of the arithmetic of natural numbers are based on Axiom P5. As an example we prove the following.

Associative law for addition

$$(m + n) + p = m + (n + p) \qquad \text{for all } m, n, p \in \mathbf{N}$$

PROOF Let $m, n \in \mathbf{N}$ be fixed but arbitrary and define

$$A = \{p \in \mathbf{N} \mid (m + n) + p = m + (n + p)\}$$

It is clear that $A \subseteq \mathbf{N}$, and since

$$(m + n) + 1 = (m + n)^* = m + n^* = m + (n + 1)$$

we have that $1 \in A$. Now suppose $p \in A$; then

$$(m + n) + p = m + (n + p)$$

Hence

$$\begin{aligned}(m + n) + p^* &= [(m + n) + p]^* = [m + (n + p)]^* \\ &= m + (n + p)^* = m + (n + p^*)\end{aligned}$$

and so $p^* \in A$. It follows from Axiom P5 that $A = \mathbf{N}$. Thus $(m + n) + p = m + (n + p)$ for every natural number p. Since m and n were arbitrary, the associative law for addition is established.

The other properties of natural numbers can be proved in a similar fashion. They are listed below, and it is recommended that the reader prove them in the order given since some of the proofs will be simpler if previously established properties are used along with the five axioms in verifying a given property.

Commutative law for addition

$$m + n = n + m \qquad \text{for all } m, n \in \mathbf{N}$$

Distributive laws

$$p \cdot (m + n) = (p \cdot m) + (p \cdot n) \qquad \text{for all } m, n, p \in \mathbf{N}$$

$$(m + n) \cdot p = (m \cdot p) + (n \cdot p) \qquad \text{for all } m, n, p \in \mathbf{N}$$

Associative law for multiplication

$$(m \cdot n) \cdot p = m \cdot (n \cdot p) \qquad \text{for all } m, n, p \in \mathbf{N}$$

Commutative law for multiplication

$$m \cdot n = n \cdot m \qquad \text{for all } m, n \in \mathbf{N}$$

Cancellation laws

$$p + m = p + n \text{ implies } m = n \qquad \text{for all } m, n, p \in \mathbf{N}$$

and

$$p \cdot m = p \cdot n \text{ implies } m = n \qquad \text{for all } m, n, p \in \mathbf{N}$$

We verify the first cancellation law and leave the second to the exercises.

PROOF Let $S(p)$ be the statement "$p + m = p + n$ implies $m = n$." If $1 + m = 1 + n$ then $m^* = n^*$, and so (by P4) $m = n$. Thus $S(1)$ is true. Suppose $S(p)$ is true; then $p + m = p + n$ implies $m = n$. Now if $p^* + m = p^* + n$ then $(1 + p) + m = (1 + p) + n$, and so $1 + (p + m) = 1 + (p + n)$. Since $S(1)$ is true, $p + m = p + n$. But $S(p)$ is true; hence $m = n$, and consequently $S(p^*)$ is true. It follows from the principle of mathematical induction that $S(p)$ is true for every natural number p; that is, $p + m = p + n$ implies $m = n$ for all $m, n, p \in \mathbf{N}$.

One of the first facts about natural numbers that we become familiar with is that some natural numbers are "larger" than others. This notion is called *order* and is introduced as follows. We define a relation in $\mathbf{N}$ called an order relation, symbolized by $<$ and read "less than." For any $m, n \in \mathbf{N}$ we write $m < n$ provided there is a natural number p such that $m + p = n$. This relation in $\mathbf{N}$ is clearly not an equivalence relation since it is neither reflexive nor symmetric. However, the transitive law is satisfied, for suppose $m < n$ and $n < p$. Then there are natural numbers q_1 and q_2 such that $m + q_1 = n$ and $n + q_2 = p$. Thus $p = n + q_2 = (m + q_1) + q_2 = m + (q_1 + q_2)$ and so $m < p$.

Now since $n + 1 = n^*$ for every natural number n, $n < n^*$; and so

$$1 < 2 < 3 < 4 < 5 < 6 < \ldots$$

The following law is fundamental and can be proved directly from the definition of the order relation in $\mathbf{N}$.

Law of trichotomy For every pair $m, n \in \mathbf{N}$ exactly one of the following holds:

(*a*) $m = n$.
(*b*) $m < n$.
(*c*) $n < m$.

We often write $m > n$ to mean $n < m$. Another relation in $\mathbf{N}$ that is frequently used is $\leq$ read "less than or equal to" and is defined by $m \leq n$ provided $m < n$ or $m = n$. This relation is both reflexive and transitive but not symmetric. It follows from the law of trichotomy that if $m \leq n$ and $n \leq m$ then $m = n$.

The next theorem, called the *well-ordering principle for* $\mathbf{N}$, is an important property which is characteristic of the set of natural numbers. This principle is used frequently in the development of the real number system (Sec. 1.4) and, as we see in the exercises, is logically equivalent to the principle of mathematical induction.

Theorem 1.2 Every nonempty subset $A \subseteq \mathbf{N}$ has a first element; that is, there is a $p \in A$ such that $p \leq a$ for every $a \in A$.

PROOF We assume that A is a nonempty subset of $\mathbf{N}$ and that A has no first element and show that this leads to a contradiction. Define $M \subseteq \mathbf{N}$ by

$$M = \{x \in \mathbf{N} \mid x < a \text{ for each } a \in A\}$$

By the law of trichotomy $M \cap A = \varnothing$. Now $1 \notin A$; otherwise 1 would surely be the first element in A. Hence $1 < a$ for each $a \in A$, and so $1 \in M$. Assume $p \in M$; then $p < a$ for each $a \in A$. If $p + 1 \in A$ then $p + 1$, which is the first natural number larger than p, would be the first element in A, in contradiction to our assumption that A has no first element. Thus $p + 1 \notin A$, and so $p + 1 < a$ for each $a \in A$. Hence $p + 1 \in M$ and by induction $M = \mathbf{N}$. But $M \cap A = \varnothing$, and so $A = \varnothing$, which is a contradiction. Therefore A must have a first element.

EXERCISES

1.12 Use the law of trichotomy to prove the cancellation law:

$$p \cdot m = p \cdot n \text{ implies } m = n \qquad \text{for all } m, n, p \in \mathbf{N}$$

1.13 Prove

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \qquad \text{for each } n \in \mathbf{N}$$

1.14 Prove

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{for each } n \in \mathbf{N}$$

1.15 Prove

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \qquad \text{for each } n \in \mathbf{N}$$