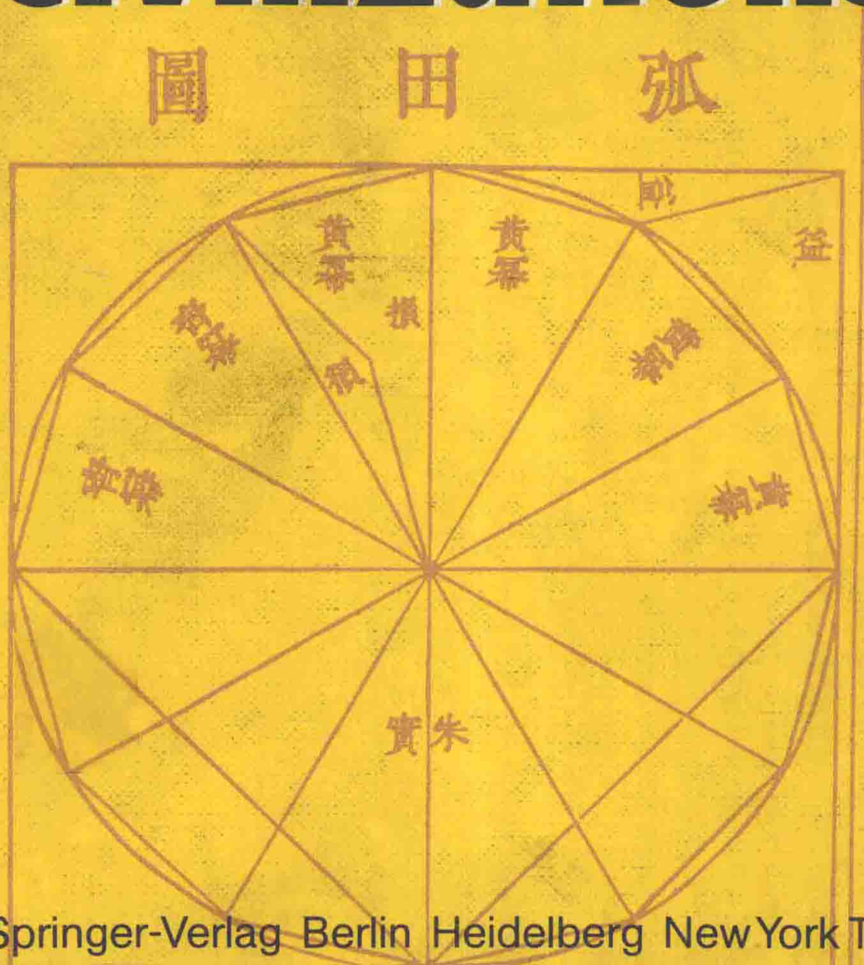
B. L. van der Waerden

Geometry and Algebra in Ancient Civilizations



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With 98 Figures

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Cover illustration

One of Tai Chen's illustrations of the "Nine Chapters of the Mathematical Art", explaining Liu Hui's method of measuring the circle (see pages 196–199). Reproduced from Joseph Needham: Science and Civilization in China, Volume 3, p. 29.

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Preface

Originally, my intention was to write a "History of Algebra", in two or three volumes. In preparing the first volume I saw that in ancient civilizations geometry and algebra cannot well be separated: more and more sections on ancient geometry were added. Hence the new title of the book: "Geometry and Algebra in Ancient Civilizations". A subsequent volume on the history of modern algebra is in preparation. It will deal mainly with field theory, Galois theory and theory of groups.

I want to express my deeply felt gratitude to all those who helped me in shaping this volume. In particular, I want to thank Donald Blackmore Wagner (Berkeley) who put at my disposal his English translation of the most interesting parts of the Chinese "Nine Chapters of the Art of Arithmetic" and of Liu Hui's commentary to this classic, and also Jacques Sesiano (Geneva), who kindly allowed me to use his translation of the recently discovered Arabic text of four books of Diophantos not extant in Greek. Warm thanks are also due to Wyllis Bandler (Colchester, England) who read my English text very carefully and suggested several improvements, and to Annemarie Fellmann (Frankfurt) and Erwin Neuenschwander (Zürich) who helped me in correcting the proof sheets. Miss Fellmann also typed the manuscript and drew the figures.

I also want to thank the editorial staff and production department of Springer-Verlag for their nice cooperation.

Zürich, February, 1983 B. L. van der Waerden

Introduction

Until quite recently, we all thought that the history of mathematics begins with Babylonian and Egyptian arithmetic, algebra, and geometry. However, three recent discoveries have changed the picture entirely.

The first of these discoveries was made by A. Seidenberg. He studied the altar constructions in the Indian Śulvasūtras and found that in these relatively ancient texts the "Theorem of Pythagoras" was used to construct a square equal in area to a given rectangle, and that this construction is just that of Euclid. From this and other facts he concluded that Babylonian algebra and geometry and Greek "geometrical algebra" and Hindu geometry are all derived from a common origin, in which altar constructions and the "Theorem of Pythagoras" played a central rôle.

Secondly I have compared the ancient Chinese collection "Nine Chapters of the Arithmetical Art" with Babylonian collections of mathematical problems and found so many similarities that the conclusion of a common pre-Babylonian source seemed unavoidable. In this source, the "Theorem of Pythagoras" must have played a central rôle as well.

The third discovery was made by A. Thom and A. S. Thom, who found that in the construction of megalithic monuments in Southern England and Scotland "Pythagorean Triangles" have been used, that is, right-angled triangles whose sides are integral multiples of a fundamental unit of length. It is well-known that a list of "Pythagorean Triples" like (3,4,5) is found in an ancient Babylonian text, and the Greek and Hindu and Chinese mathematicians also knew how to find such triples.

Combining these three discoveries, I have ventured a tentative reconstruction of a mathematical science which must have existed in the Neolithic Age, say between 3000 and 2500 B.C., and spread from Central Europe to Great Britain, to the Near East, to India, and to China. By far the best account of this mathematical science is found in Chinese texts. My ideas concerning this ancient science will be explained in Chapters 1 and 2.

The Greeks had some knowledge of this ancient science, but they transformed it completely, creating a deductive science based on definitions, postulates and axioms. Yet several traces of pre-Babylonian geometry and algebra can be discerned in the work of Euclid and Diophantos and in popular Greek mathematics. This will be shown in Chapters 3, 4 and 6.

In the treatises of Hindu astronomers like Āryabhaṭa and Brahmagupta, who lived in the sixth and seventh century A.D., we find methods to solve Diophantine equations such as

$$ax + c = by$$

and

$$x^2 = Dy^2 + 1.$$

These methods are based on the Euclidean algorithm. In Chapter 5 I shall give an account of these methods and discuss their relation to Greek science.

Chapter 7 deals with the work of the excellent Chinese geometer Liu Hui (third century A. D.) and with some mathematical passages in the work of the great Indian astronomer Āryabhaṭa (sixth century). It seems to me that both were influenced by the work of Greek geometers and astronomers like Archimedes and Apollonios. In particular I shall discuss Liu Hui's measurement of the circle and Āryabhaṭa's trigonometry.

Table of Contents

Chapter 1

Pythagorean Triangles

Part A

Written Sources

Fundamental Notions

A *Pythagorean Triangle* is a right-angled triangle in which the three sides are proportional to integers x, y, and z. According to the "Theorem of Pythagoras", the integers must satisfy the equation

$$x^2+y^2=z^2. \tag{1}$$

A *Pythagorean Triple* is a triple of integers (x,y,z) satisfying (1). The triple is *primitive* if x,y,z have no common factor.

In a primitive Pythagorean triple one of the numbers x and y must be odd and the other even, because if both are even, the triple is not primitive, and if both are odd, the sum x^2+y^2 is a number of the form $4n+2$ and cannot be a square.

The construction of primitive Pythagorean triples is a simple arithmetical problem. One can start with any integer x and solve the equation

$$(z-y)(z+y)=x^2 \tag{2}$$

for y and z. If one starts with an odd number $x=st$, one can satisfy the equation (2) by taking

$$\begin{aligned} z+y&=s^2 \qquad & z&=\tfrac{1}{2}(s^2+t^2)\\ z-y&=t^2 \qquad & y&=\tfrac{1}{2}(s^2-t^2). \end{aligned} \tag{3}$$

As we shall see presently, this method of solution was used in a Chinese text from the Han-period (about -200 to $+220$). On the other hand, if one starts with an even integer $x=2pq$, one can satisfy (2) by taking

$$(4)\qquad \begin{aligned} z+y&=2p^2 & z&=p^2+q^2\\ z-y&=2q^2 & y&=p^2-q^2. \end{aligned}$$

This solution was used by Diophantos of Alexandria. In book VI of his Arithmetica the first problem reads:

Problem 1. To find a right-angled triangle such that the hypotenuse minus each of the other sides gives a cube.

The solution begins thus:

Let the triangle be formed with the aid of two numbers, say s and 3. The hypotenuse is then s^2+9, the height $6s$ and the base s^2-9.

In our text-books of elementary number theory it is proved that all primitive Pythagorean triples can be obtained from the formulae (4). Just so, one can prove that all primitive Pythagorean triples can be obtained from (3). In fact, (4) can be obtained from (3) by substituting

$$s=p+q\,, \qquad t=p-q$$

and interchanging x and y. So the two methods for finding Pythagorean triples, the Chinese and the Greek method, are equivalent.

The Text Plimpton 322

Long before Diophantos, the Babylonians already knew how to calculate Pythagorean triples. To see this, let us consider the cuneiform text "Plimpton 322", written under the dynasty of Hammurabi, and published by Neugebauer and Sachs in their book "Mathematical Cuneiform Texts" (New Haven 1945). It has been discussed by Peter Huber (l'Enseignement mathématique 3, p. 19), by Derek de Solla Price (Centaurus 10, p. 219–231), and by E. M. Bruins (Physis 9, p. 373–392). Here I shall use only those facts and interpretations on which all authors agree.

The text is the right-hand part of a larger tablet containing several columns of numbers. The last column contains nothing but the numbers 1, 2,...,15. The preceding columns refer, according to the legend at the head of the columns, to the "width" and the "diagonal" (of a rectangle). I shall denote the "width" by y and the "diagonal" by z. If one calculates z^2-y^2, one finds that it is in all the cases the square of an integer x containing only factors 2, 3 and 5. Such "regular numbers" x play a special role in the sexagesimal number system, because the reciprocals x^{-1} can be written as finite sexagesimal fractions. The Babylonians had tables of reciprocals of these "regular numbers".

A few errors in the text have been corrected already by Neugebauer and Sachs. After these corrections, the 15 triples (x,y,z) are:

x	y	z
2, 0	1,59	2,49
57,36	56, 7	1,20,25
1,20, 0	1,16,41	1,50,49
3,45, 0	3,31,49	5, 9, 1
1,12	1, 5	1,37
6, 0	5,19	8, 1
45, 0	38,11	59, 1
16, 0	13,19	20,49
10, 0	8, 1	12,49
1,48, 0	1,22,41	2,16, 1
1, 0	45	1,15
40, 0	27,59	48,49
4, 0	2,41	4,49
45, 0	29,31	53,49
1,30	56	1,46

The numbers are written in the sexagesimal system based on powers of 60. For instance, the number 2,49 in the first line means

$$2 \times 60 + 49 = 169 .$$

In the text, the two columns headed y and z are preceded by a column representing either

$$A = (y/x)^2$$

or

$$1 + A = (x^2 + y^2)/x^2 = (z/x)^2 .$$

For instance, in the first line, we have

$$x = 2,0 \quad \text{and} \quad y = 1{,}59$$

hence

$$y/x = 0;59{,}30$$

which means $59/60 + 30/60^2$. We now have

$$A = (y/x)^2 = 0;59,\ 0{,}15$$

and

$$1 + A \qquad = 1;59,\ 0{,}15 .$$

We cannot distinguish between the two possibilities A and $1+A$, because in the first column the initial digits 1, if they ever existed, are broken off.

All commentators agree that the preserved columns were probably preceded by at least three columns, in which the quotients

$$y/x=v \quad \text{and} \quad z/x=w$$

and the height x were recorded. This means: the Babylonians first calculated a Pythagorean triple $(1,v,w)$ satisfying the equation

$$1+v^2=w^2 \tag{5}$$

and next multiplied the triple $(1,v,w)$ by a suitable number x in order to obtain integer triples (x,y,z).

There are three arguments in favour of this hypothesis:

First, in one case (line 11 of the tablet) the triple $(1,v,w)$ would be

$$1, \quad v=0;45, \quad w=1;15.$$

A multiplication by 4 would have yielded the well-known Pythagorean triple (4,3,5). Instead of performing this easy multiplication, the scribe left the triple $(1,v,w)$ as it stands and wrote

$$1, \quad 45, \quad 1,15.$$

Note that in the Babylonian notation we cannot distinguish between $1,15=75$ and $1;15=1\ 15/60$.

The *second* argument is: the Babylonians had a column

$$A=v^2 \quad (\text{or } 1+A=w^2).$$

This column was probably derived from an earlier column v (or w) by squaring.

The *third* argument is: it is easier to solve the equation (5) than the equation (1), for (5) can be written as

$$(w+v)(w-v)=1. \tag{6}$$

In all 15 cases, $w+v$ is a regular sexagesimal number. If this number is called d, we have

$$w+v=d$$
$$w-v=d^{-1}$$

from which v and w can be solved easily.

It is possible that the Babylonians took d and d^{-1} directly from a table of reciprocals. It is also possible that they put

$$d=p/q=pq^{-1}$$

and

$$d^{-1}=q/p=qp^{-1}$$

as Neugebauer and Sachs supposed. I cannot decide between these possibilities.

If the Pythagorean triples of Plimpton 322 were computed by this method, the Babylonian scribes must have known the "Theorem of Pythagoras" as well as the algebraic identity

$$z^2 - y^2 = (z+y)(z-y). \tag{7}$$

Both conclusions are confirmed by other Babylonian texts. For instance, in the text BM 85 196 the hypotenuse $z=30$ and the height $y=24$ of a right-angled triangle are given, and the base $x=18$ is calculated as the square root of z^2-y^2 (see my Science Awakening I, p. 76). The identity (7) was used in many Babylonian problem solutions, as we shall see in Chapter 2. Note: the signature BM means British Museum.

All in all, we may safely conclude that the Babylonians calculated Pythagorean triples (x,y,z) by first assuming x to be 1, and next multiplying the triple $(1,v,w)$ by a suitable integer.

Can we find similar methods in other civilizations? Yes, we can.

A Chinese Method

The Chinese collection of mathematical problems "Nine Chapters on the Mathematical Art"[1], written during the Han-period (circa −200 to +220), contains in Chapter 9 a sequence of problems on right-angled triangles. The problems will be discussed in greater detail in Chapter 2. For the moment, I only note that all solutions are based on the "Theorem of Pythagoras". For instance, in Problem 1, x and y are given, and z is computed as the square root of x^2+y^2.

In the 16 problems of Chapter 9, the following Pythagorean triangles occur:

3	4	5
5	12	13
8	15	17
7	24	25
20	21	29.

How did the author of the problems find these triangles? Light upon this question is shed by the very remarkable problem 14. The following translation (with explanations in parentheses) is due to Donald Blackmore Wagner, who kindly put at my disposal his translations of several problems of Chapter 9. Problem 14 reads:

1 German translation by Kurt Vogel: Neun Bücher arithmetischer Technik (Vieweg, Braunschweig 1968).

Two persons are standing together. The proportion of *A's* walking is 7, and the proportion of *B's* walking is 3. (That is, the ratio of their walking speeds is $a:b=7:3$).

B walks east. *A* walks south $10pu$, then walks diagonally (roughly) northeast, and meets *B*. How far do *A* and *B* walk?

Anwer: *B* walks $10\frac{1}{2}pu$. A walks diagonally $14\frac{1}{2}pu$.

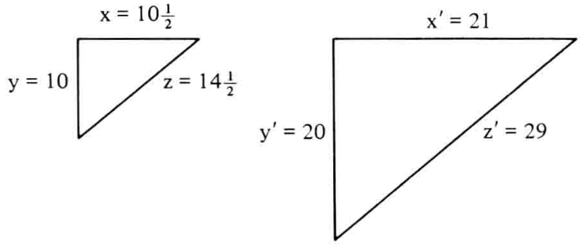

Fig. 1. A Chinese walking problem

To make things clearer, I have drawn two similar rightangled triangles $(10,10\frac{1}{2},14\frac{1}{2})$ and (20,21,29). The Chinese text has no drawings. The text explains the method of solution thus:

Method: Multiply 7 by itself; also multiply 3 by itself. Add (the products) and halve. This is the proportion of *A* walking diagonally ($z'=29$).

Subtract from the proportion (of *A* walking diagonally) the product of 3 by itself. The difference is the proportion of walking south ($y'=29-9=20$).

Let the product of 3 and 7 be the proportion of *B* walking east ($x'=21$).

Set up the $10\,pu$ walked south, and multiply by the proportion of *A* walking diagonally (result $yz'=10\times 29=290$). Again set up $10\,pu$ and multiply by the proportion of *B* walking east (result $yx'=10\times 21=210$). Let each (result) be a dividend. Dividing the dividends by the proportion of walking south ($y'=20$) gives in each case the number (of *pu*) walked.

In the statement of the problem as well as in the solution, a mathematical terminus is used which Wagner has translated as "proportion". The expression "the proportion of *A's* walking is 7, and the proportion of *B's* walking is 3" means: the distances traversed by *A* and *B* in the same time are in the ratio of 7 to 3. In Fig. 1, this means

$$z+y=(7/3)x. \tag{8}$$

We also are given

$$y=10. \tag{9}$$

The method by which the Chinese author calculates the sides *x, y, z* of the required triangle is very remarkable. He first finds a Pythagorean triple (x', y', z') satisfying the condition (8). He calls x' "the proportion of *B* walking east", and similarly for y' and z'. That is, the author knew that he had to make *x, y, z* proportional to x', y', z' in order to reach his aim. The latter triple was calculated as follows, starting with the given numbers 7 and 3: