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Representations of Real Reductive Lie Groups

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Preface

Since this manuscript was completed in August, 1980, a great deal of progress has been made on some of the topics treated. Working independently, Brylinski-Kashiwara and Beilinson-Bernstein have established Conjecture 2.1.7 on the composition series of Verma modules. Lusztig and I have used the work of Beilinson and Bernstein to prove Conjectures 2.2.12 and 2.3.11 on Harish-Chandra modules, computing all composition series in the case of integral infinitesimal character. [This settles Problems 9 and 10 of Section 9.7.] The non-integral case remains open. The proofs rely on the Weil conjectures and the Deligne-Goresky-MacPherson intersection homology theory. It is not yet clear how these developments will affect the material presented in this book -- they do not provide any obvious substantial simplifications, although in some cases they suggest significant improvements in the formulation of the theory.

I would like to thank Janet Ellis for typing and proofreading a long and messy manuscript with great care and skill in the midst of many other responsibilities. I have the dubious satisfaction of knowing that no errors remain which are not of my own devising.

Cambridge, Massachusetts

April, 1981

Introduction

This book is a survey of some recent work on the (non-unitary) infinite dimensional representations of a real reductive Lie group G. There are three major topics. The first is Langlands' theorem on the classification and realization of the irreducible representations of G (Theorem 6.5.12). They are described in terms of certain "standard representations" (Definition 6.5.2), which generalize the principal series and are sometimes reducible. The second topic is the reducibility of these standard representations. The main results (Theorem 8.6.6 and Proposition 8.7.6) are due to B. Speh and the author; they are not quite decisive. The third topic is a conjecture which describes explicitly the decomposition of standard representations into irreducible representations -- or, equivalently, the Harish-Chandra characters of the irreducible representations. This generalizes a conjecture of Kazhdan and Lusztig for Verma modules, and is described in Section 9.6.

Since the theory of non-unitary representations of G was created by Harish-Chandra essentially as a means to study unitary representations, some apology might seem to be required for a book in which unitary representations play almost no part. The first explanation for this omission is simply a lack of space. The first two topics at least are very important for recent work on unitary representations. For example, the study of unitary representations with non-zero continuous cohomology (see [3]) has been advanced by the algebraic study of certain representations which are still not known to be unitary. (They are included in the conjecturally unitary representations of (6.5.17) below.) The theory of "complementary series" of unitary representations depends on (among other things) an understanding of the reducibility of standard representations. Thus Theorem 8.6.6 provides a large number of unitary representations, and a proof of the conjecture of Section 9.6 would provide even more.

The real explanation, however, is that non-unitary representation theory is interesting enough not to require any such justification. Such a claim has to be supported in the text and not in the introduction; but Chapter 2 attempts to describe the nature of the main results without too much technical clutter.

In a little more detail, the book is organized as follows. The reader is assumed to be quite familiar with the structure and finite dimensional representation theory of complex reductive Lie algebras; this is really a prerequisite even for understanding the statements of most of the results. Logically, the book also depends on Harish-Chandra's basic theory relating group representations and Lie algebra representations ([12]) and on his subquotient theorem ([13]). These topics are treated in [49] or [50], and the results are summarized in Sections 0.3 and 4.1 of this book. Since the ideas needed to prove them will not be used here, the reader who is willing to take them on faith should have little difficulty. The first three sections of Chapter 1 summarize, with some proofs, the representation theory of $SL(2, \mathbb{R})$. This is intended to provide examples as guides to the rather abstract and technical treatment of the general case. Some results are proved in general by reduction to $SL(2, \mathbb{R})$, and the necessary special cases of these are discussed in the rest of Chapter 1.

Chapter 2 contains a detailed statement of the Langlands classification of irreducible representations of G in a special case; and a geometric formulation of the conjecture of Section 9.6 on composition series of standard representations. (The two formulations are not known to be equivalent.) The entire chapter is meant as an extended introduction to the rest of the book.

The main technical tool used here to study representations is Lie algebra cohomology (of the nil radical of a parabolic subalgebra, with coefficients in a representation). Chapter 3 contains two fundamental theorems in that subject: the Casselman-Osborne theorem relating cohomology and the center of the enveloping algebra, and Kostant's formulation of the Bott-Borel-Weil theorem.

Chapter 4 discusses that part of the classification of irreducibles which can be obtained from ordinary principal series representations. In addition to more standard intertwining operator techniques, it uses the Bernstein-Gelfand-Gelfand theory of fine representations (see [2]); a detailed account of this theory is given in Section 4.3.

Chapters 5 and 6 complete the classification of irreducible representations. The method is discussed in some detail in the introductions to those chapters. Essentially it is a generalization of the highest weight theory of finite dimensional representations, with highest weight vectors replaced by a more general kind of cohomology classes for the representation. The main problem is to find a construction of representations, generalizing induction, which is nicely related to cohomology. This was done (for reductive groups) by G. Zuckerman (Definition 6.3.1). This book uses only one special case of this definition, in addition to ordinary induction. It seems likely that one can do much more with the idea.

Chapter 7 is devoted to the Jantzen-Zuckerman "translation principle" and related matters. This says that all irreducible representations come in nice families (like the principal series, or finite dimensional representations). Many results can therefore be proved by reducing to the case when the representation is in "general position" in some sense. This is helpful technically, but the translation principle plays an even more fundamental role. Roughly speaking, it provides a connection between the structure of irreducible representations, and the combinatorial structure of the Weyl group. This is discussed more carefully in Section 7.3; the key result is Theorem 7.3.16.

In Chapter 8, the basic theorem on reducibility of standard representations is proved. This is in principle a trivial consequence of the results of Chapter 7, but requires some messy calculations. (For example, we need the Hecht-Schmid character identities for discrete series from [15].) Exactly the same ideas, with the judicious addition of a technical conjecture (Conjecture 7.3.25), lead to the algorithm for computing composition series; this is the content of Chapter 9.

The book differs from the existing literature in several ways. Most importantly, the standard representations are not constructed by Langlands' method (that is, ordinary induction from discrete series). We use instead Zuckerman's "cohomological" induction from principal series. This gives isomorphic standard representations (Theorem 6.6.15), but that fact is not proved in the text (or used). As far as the classification itself goes, this choice is simply a matter of taste. However, the conjecture on composition series seems to be comprehensible only in the realization we have given. (There is an obvious way to try to use ordinary induction, but then any simple analogue of the critical Theorem 9.5.1 is false.) This is not to say that Zuckerman's realization is better; for analytic problems, it is Zuckerman's method which encounters obstacles.

There are many less substantive changes. What I had perceived as the main theorem of [42], which relates cohomology and $U(\mathfrak{g})^K$, does not appear here; the argument has been rearranged slightly to eliminate the need for it. The definition of lowest K-type in [42] (Definition 5.4.18 below) has been replaced by a much more technical one (Definition 5.4.1); they are equivalent for irreducibles but not for general $(\mathfrak{g},K)$ modules. The advantage of the new definition is that a number of subsequent technical arguments become much simpler when it is used.

The proof of the Knapp-Stein reducibility theorem (Corollary 4.4.11) is new, avoiding both the rather delicate analysis used in [26] and the long case-by-case computation in the unpublished second part of [42].

The main result about translation "across a wall" is Theorem 7.3.16; it is (or was) the only non-formal part of the proof of the theorem on reducibility of standard representations. About one-third of [39] is devoted to a proof of it, which might charitably be described as sketchy. A second proof was given in [45], which was fairly short, but used Duflo's main theorem on primitive ideals (see [8]). The proof given here is entirely trivial; but the result is labelled as a theorem in memory of [39].

Each chapter begins with an introduction; these provide a more detailed guide to the main results. Section 9.7 summarizes some open problems.

This book is based on lectures given at MIT during the 1979-80 academic year. I would like to thank those who attended for helpful comments and dogged perseverance. Much of Chapter 6 is unpublished work of G. Zuckerman. I thank him for explaining it to me, and allowing it to appear here. J. Vargas provided a list of errors in the first draft, which was very helpful. Many people have pointed out particular errors and shortcomings; I apologize for those which undoubtedly remain.

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Chapter 0. Preliminaries

General references for this chapter are [40] and [50].

§1. Assumptions on G

Notation 0.1.1 Suppose H is a real Lie group. Write

H_0 = identity component of H;

$\mathfrak{h}_0$ = Lie algebra of H;

Ad: $H \to \mathrm{End}(\mathfrak{h}_0)$, the adjoint representation of H;

$\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$, the complexification of $\mathfrak{h}$;

$U(\mathfrak{h})$ = universal enveloping algebra of $\mathfrak{h}$.

$\mathfrak{h}$ is given the complex structure bar, defined by

$$\overline{X + iY} = X - iY \qquad (X, Y \in \mathfrak{h}_0).$$

This notation will be applied to groups denoted by other Roman letters in the same way without comment.

By a real reductive linear group, we will mean a real Lie group G (not necessarily connected), a maximal compact subgroup K of G, and an involution θ of $\mathfrak{g}_0$, satisfying the following conditions.

(0.1.2)

a) $\mathfrak{g}_0$ is a real reductive Lie algebra;

b) If $g \in G$, the automorphism Ad(g) of $\mathfrak{g}$ is inner (for the corresponding complex connected group);

c) The fixed point set of θ is $\mathfrak{k}_0$;

d) Write $\mathfrak{p}_0$ for the -1 eigenspace of θ; then the map $K \times \mathfrak{p}_0 \to G$, $(k, X) \to k \cdot \exp(X)$ is a diffeomorphism;

e) G has a faithful finite dimensional representation;

f) Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be a Cartan subalgebra, and let H be be the centralizer of $\mathfrak{h}_0$ in G. Then H is abelian.

(Notice that (d) forces G to have only finitely many components.)

<u>Throughout this book, G will denote a real reductive linear group</u>. We will from time to time assume that G satisfies additional conditions, but these at least must always be met. Conditions (a) - (d) define Harish-Chandra's category of real reductive groups (cf. [40], §5) and require no further justification here; we will use inductive arguments which lead from connected groups to disconnected ones. Assumption (e) is in some sense only a convenience -- most of the results we obtain can be gotten without it, although sometimes this requires more work and a less satisfactory formulation of the theorems. However, one of our main goals is the formulation of the Kazhdan-Lusztig conjectures discussed in the introduction; and this has not been carried out for non-linear groups. (The problems do not seem to be very deep, but they are quite messy.) Assumption (f) is included chiefly to make the Knapp-Stein "commutativity of intertwining operators" theorem (Corollary 6.5.14) hold. (The simplest case where it fails has $|G/G_0| = 4$, $G_0 = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$; (a) - (e) are satisfied, but (f) is not.)

<u>Definition 0.1.3</u> Make θ an involution of G by setting

$$\theta(k \cdot \exp(X)) = k \cdot \exp(-X) \quad (X \in \mathfrak{p}_0,\ k \in K) .$$

We call θ the <u>Cartan involution</u> of G or $\mathfrak{g}_0$.

Example 0.1.4

a) G = connected real linear semisimple group;

b) $G = GL(n, \mathbb{R})$, $K = O(n)$, $\theta(g) = {}^t g^{-1}$;

c) G = real points of a reductive algebraic group defined over $\mathbb{R}$;

d) If $G_{\mathbb{C}}$ is a complex semisimple Lie algebra, and $\mathfrak{g}_0$ is a real form of its Lie algebra $\mathfrak{g}$, then

$$G = \text{normalizer in } G_{\mathbb{C}} \text{ of } \mathfrak{g}_0;$$

e) $G = SL(2,\mathbb{R}) \cup SL(2,\mathbb{R}) \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$;

f) Suppose G is a real reductive linear group, and H is a θ-stable abelian subgroup. Then the centralizer G^H of H in G is a real reductive linear group.

Of course, these examples overlap enormously. We will make constant use of example (f); its (easy) verification is left to the reader.

Example (b) illustrates an annoying technical problem: K need not be a real reductive group. The problem is that the orthogonal group O(n) does not satisfy (0.1.2)(b). Thus the Cartan-Weyl highest weight theory does not apply directly to K; we do not have a priori a perfect grasp on $\hat{K}$. This is circumvented by using the relation between K and G; a description of $\hat{K}$ is given in Section 5.1.

Definition 0.1.5 A Cartan subgroup of G is the centralizer in G of a Cartan subalgebra of $\mathfrak{g}_0$. A parabolic subgroup of G is the normalizer in G of a parabolic subalgebra of $\mathfrak{g}_0$.