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Hans Schoutens

The Use of Ultraproducts in Commutative Algebra

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$$\mathbb{C} \cong \operatorname*{ulim}_{p \to \infty} \mathbb{F}_p^{alg}$$

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To my mother, Jose Van Passel,
for giving me wisdom;
to my father, Louis Schoutens,
for giving me knowledge;
to my teacher, Pierre Gevers,
for giving me the love for mathematics;
to my mentor, Jan Denef,
for giving me inspiration;
and to my one and true love,
Parvaneh Pourshariati,
for giving me purpose.

Contents

Chapter 1
Introduction

Unbeknownst to the majority of algebraists, ultraproducts have been around in model-theory for more than half a century, since their first appearance in a paper by Łoś ([65]), although the construction goes even further back, to work of Skolem in 1938 on non-standard models of Peano arithmetic. Through Kochen's seminal paper [61] and his joint work [9] with Ax, ultraproducts also found their way into algebra. They did not leave a lasting impression on the algebraic community though, shunned perhaps because there were conceived as non-algebraic, belonging to the alien universe of set-theory and non-standard arithmetic, a universe in which most mathematicians did not, and still do not feel too comfortable.

The present book intends to debunk this common perception of ultraproducts: when applied to algebraic objects, their construction is quite natural, yet very powerful, and requires hardly any knowledge of model-theory. In particular, when applied to a collection of rings A_w, where w runs over some infinite index set W, the construction is entirely algebraic: the ultraproduct of the A_w is realized as a certain residue ring of the Cartesian product $A_\infty := \prod A_w$ modulo the so-called *null-ideal* (see below). Any ring arising in this way will be denoted $A_\natural$, and called an *ultra-ring*;[1] and the A_w are then called *approximations* of this ultra-ring. As this terminology suggests, we may think of ultraproducts as certain kinds of limits. This is the perspective of [102], which I will not discuss in these notes.

Whereas the classical Cartesian product performs a parallel computation, so to speak, within each A_w, the ultraproduct, on the other hand, computes things generically: elements in the ultraproduct $A_\natural$ satisfy certain algebraic relations if and only if their corresponding entries satisfy the same relations in the approximations A_w *with probability one*. To make this latter condition explicit, an ostensibly extrinsic component has to be introduced: we must impose some (degenerated) probability measure on the index set W of the family. The classical way is to choose a (non-principal) ultrafilter on W, and then say that an *event holds with probability one* (or, more informally, *almost always*) if the set of indices for which it holds belongs to the ultrafilter. Fortunately, the dependence on the choice of

[1] For the rather unorthodox notation, see below.

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ultrafilter/probability measure turns out to be, for all our intents and purposes, irrelevant, and so ultraproducts behave almost as if they were intrinsically defined.[2]

Once we have chosen a (non-principal) ultrafilter, we can define the ultraproduct $A_\natural$ as the residue ring of the Cartesian product $A_\infty := \prod_w A_w$ modulo the null-ideal of *almost zero elements*, that is to say, those elements in the product almost all of whose entries are zero. However, we can make this construction entirely algebraic, without having to rely on an ultrafilter/probability measure (although the latter perspective is more useful when we have to prove things about ultraproducts). Namely, A_∞ carries naturally the structure of a $\mathbb{Z}_\infty$-algebra, where $\mathbb{Z}_\infty$ is the corresponding Cartesian power of the ring of integers $\mathbb{Z}$. Given any minimal prime ideal $\mathfrak{P}$ in $\mathbb{Z}_\infty$, the base change $A_\natural := A_\infty/\mathfrak{P}A_\infty$ is an ultra-ring (with corresponding null-ideal $\mathfrak{P}A_\infty$). Moreover, all possible ultraproducts of the A_w arise in this way (see §2.5). Principal null-ideals, corresponding to principal ultrafilters, have one of the A_w as residue rings, and therefore are of little use. Hence from now on, when talking about ultra-rings, we always assume that the null-ideal is not principal—it follows that it is then infinitely generated—and this is equivalent with the ultrafilter containing all co-finite subsets, and also with $\mathfrak{P}$ containing the direct sum ideal $\bigoplus \mathbb{Z}$. Perhaps even more surprisingly familiar is the alternative definition given in §2.6 (communicated to me by Macintyre): an ultra-ring is simply a stalk at a point x of a sheaf of rings on a Boolean scheme, where a scheme is called *Boolean* if each residue field is isomorphic to $\mathbb{F}_2$ (and the null-ideal is non-principal if and only if the prime ideal of x is infinitely generated).

I already alluded to the main property of ultraproducts: they have the same (first-order) properties than almost all their approximations A_w; this is known to model-theorists as Łoś' Theorem. Although it may not always be easy to determine whether a property carries over, that is to say, is first-order, this is the case if it is expressible in arithmetic terms. *Arithmetical* here refers to algebraic formulas between ring elements, 'first-order objects,' but not between 'higher-order objects,' like ideals or modules. For instance, properties such as being a domain, reduced, normal, local, or Henselian, are easily seen to be preserved. Among those that do not carry over, is, unfortunately, the Noetherian property. Ultra-rings, therefore, are hardly ever Noetherian; the ultraproduct construction takes us outside our category! In particular, tools from commutative algebra seem no longer applicable. However, as we will show, there is still an awful lot, especially in the local case, that does carry through, with a few minor adaptations of the definitions. In fact, we will introduce two variant constructions that are designed to overcome altogether this obstacle. I have termed these *chromatic products*, for they, too, are denoted using musical notation: the *protoproduct* $A_\flat$, and the *cataproduct* $A_\sharp$. The latter is defined as soon all A_w are Noetherian local rings of bounded embedding dimension (that is to say, whose maximal ideal is gener-

[2] This does not mean that ultraproducts of the same rings, but with respect to different ultrafilters, are necessarily isomorphic.

ated by n elements, for some n independent from w). Its main advantage over the ultraproduct itself, of which it is a further residue ring, is that a cataproduct is always Noetherian and complete. To define protoproducts, we need some additional data on the approximations, namely, some uniform grading, analogous to polynomial degree. Although protoproducts do not need to be Noetherian, they often are. In case both are defined, we get a *chromatic scale of homomorphisms* $A_\flat \to A_\natural \to A_\sharp$.

However, as we shall see, it is in combination with certain flatness results that ultraproducts, and more generally chromatic products, acquire their real power. Already in their 1984 paper [86], Schmidt and van den Dries observed how a certain flatness property of ultraproducts, discovered five years prior to this by van den Dries in [25], translates into the existence of uniform bounds in polynomial rings (see our discussion in §4.2). This paper was soon followed by others exploiting this new method: [11,23,84]. The former two papers brought in a third theme that we will encounter in this book on occasion: Artin Approximation (see §7.1). So germane to almost every single application of ultraproducts is flatness, that I have devoted a separate chapter, Chapter 3, to it. It contains several flatness results, old and new,[3] that will be of use later in the book. Prior to this chapter, I introduce first our main protagonist, the ultra-ring, and prove some elementary facts. Noteworthy is a model-theoretic version of the Lefschetz Principle, Theorem 2.4.3, which will provide the basis of most transfer results from positive to zero characteristic: we may realize the field of complex numbers as an ultraproduct of fields of positive characteristic!

The subsequent chapters—except for Chapter 5, which is a brief survey on classical tight closure theory—then contain deeper results and properties of ultra-rings. Since an ultraproduct averages or captures the generic behavior of its approximations, it should not come as a surprise that as a tool, it is particularly well suited to derive uniformity results. This is done in Chapter 4, whose material is both thematically and chronologically the closest to its above mentioned paradigmatic forebear [86]. A second, more profound application of the method to commutative algebra is described in Chapters 6 and 7: we use ultraproducts to give an alternative treatment of tight closure theory in characteristic zero. Tight closure theory, introduced by Hochster and Huneke in an impressive array of beautiful articles—[47,48,50,53,51], to name only a few—is an extremely powerful tool, which relies heavily on the algebraicity of the Frobenius in positive characteristic, and as such is primarily a positive characteristic tool. Without going into details (these can be found in Chapter 5), one associates, using the p-th power Frobenius homomorphisms, to any ideal $\mathfrak{a}$ in a ring of characteristic $p > 0$, its *tight closure* $\mathfrak{a}^*$, an overideal contained in the integral closure of $\mathfrak{a}$, but often much closer or "tighter" to the original $\mathfrak{a}$. What really attracted people to the method was not only the

[3] Some of the well-known criteria are given here with a new proof; see, for instance, §3.3.6 on the Local Flatness Criterium.

apparent ease with which deep, known results could be reproved, but also its new, and sometimes unexpected applications, both in commutative algebra and algebraic geometry, derived almost all by means of fairly elementary arguments.

Although essentially a positive characteristic method, its authors also conceived of tight closure theory in characteristic zero in [54], by a generic reduction to positive characteristic. In fact, this reduction method, using Artin Approximation, as well as the method in positive characteristic itself were both inspired by the equally impressive work of Peskine and Szpiro [75] on Intersection Conjectures, and Hochster's own early work on big Cohen-Macaulay modules ([56]) and homological conjectures ([43,44]). However, to develop the method in characteristic zero some extremely deep results on Artin Approximation[4] were required, and the elegance of the positive characteristic method was entirely lost. No wonder! In characteristic zero, there is no Frobenius, nor any other algebraic endomorphism that could take over its role. To the rescue, however, come our ultraproducts. Keeping in mind that an ultraproduct is some kind of averaging process, it follows that the ultraproduct of rings of different positive characteristic is an ultra-ring of characteristic zero, for which reason we call it a *Lefschetz ring*. Furthermore, the ultraproduct of the corresponding Frobenius maps—one of the many advantages of ultraproducts, they can be taken of almost anything!—yields an *ultra-Frobenius* on this Lefschetz ring. Notwithstanding that it is no longer a power map, this ultra-Frobenius can easily fulfill the role played by the Frobenius in the positive characteristic theory. The key observation now is that many rings of characteristic zero—for instance, all Noetherian local rings, and all rings of finite type over a field—embed in a Lefschetz ring via a faithfully flat homomorphism. Flatness is essential here: it guarantees that the embedded ring preserves its ideal structure within the Lefschetz ring, which makes it possible to define the tight closure of its ideals inside that larger ring. In this manner, we can restore the elegant arguments from the positive characteristic theory, and prove the same results with the same elegant arguments as before. The present theory of characteristic zero tight closure is the easiest to develop for rings of finite type over an algebraically closed field, and this is explained in Chapter 6. The general local case is more complicated, and does require some further results on Artin Approximation, although far less deep than the ones Hochster and Huneke need for their theory. In fact, conversely, one can deduce certain Artin Approximation results from the fact that any Noetherian local ring has a faithfully flat Lefschetz extension (see in particular §7.1.4). Chapter 7 only develops the parts necessary to derive all the desired applications; for a more thorough treatment, one can consult [6].

In a parallel development, Hochster and Huneke's work on tight closure also led them to their discovery of canonically defined, big balanced Cohen-Macaulay algebras in positive characteristic: any system of parameters in a excellent local domain of positive characteristic becomes a regular sequence in the absolute integral closure of the ring. The same statement is plainly false in characteristic

[4] The controversy initially shrouding these results is a tale on its own.

zero, and the authors had to circumvent this obstruction again using complicated reduction techniques. Using ultraproducts, one constructs, quite canonically, big balanced Cohen-Macaulay algebras in characteristic zero simply by (faithfully flatly) embedding the ring inside a Lefschetz ring and then taking the ultraproduct of the absolute integral closures of the positive characteristic approximations of this Lefschetz ring. With aid of these new techniques, I was able to give new characterizations of rational and log-terminal singularities. Furthermore, exploiting the canonical properties of the ultra-Frobenius, I succeeded in settling some of the conjectures that hitherto had remained impervious to tight closure methods. All these results, unfortunately, fall outside the scope of this book, and the reader is referred to the articles [94, 95, 99], or to the survey paper [100].

The next two chapters, Chapter 8 on cataproducts, and Chapter 9 on protoproducts, develop the theory of the chromatic products mentioned already above. Most of the applications are on uniform bounds. For instance, we discuss some of the characterizations from [101] of several ring-theoretic properties of Noetherian local rings, such as being analytically unramified, Cohen-Macaulay, unmixed, etc., in terms of uniform behavior of two particular ring-invariants: order (with respect to the maximal ideal) and *degree*. This latter invariant measures to which extent an element is a parameter of the ring, and is a spin-off of our analysis of the dimension theory for ultra-rings (Krull dimension is one of the many invariants that are not preserved under ultraproducts, requiring a different approach via systems of parameters). Protoproducts, on the other hand, are designed to study rings with a generalized grading, called *proto-grading*, and most applications are again on uniform bounds in terms of these. This is in essence a formalization of the method coming out of the aforementioned [86].

In the last chapter, we discuss some open problems, commonly known as *homological conjectures*. Whereas these are now all settled in equal characteristic, either by the older methods, or by the recent tight closure methods, the case when the Noetherian local ring has different characteristic than its residue field, the *mixed characteristic case*, is for the most part still wide open (other than the recent breakthrough in dimension three by Heitman [40] and Hochster [46]). We will settle some of them, at least *asymptotically*, meaning, for large enough residual characteristic. This is still far from a complete solution, and our asymptotic results would only gain considerable interest if the actual conjectures turned out to be false. The method is inspired by Ax and Kochen's solution of a problem posed by Artin about C_2-fields, historically the first application of ultraproducts outside logic (see §10.1.2). Their main result, generalized latter by Eršhov ([29, 30]), is that an ultraproduct of mixed characteristic discrete valuation rings of different residual characteristics is isomorphic to an ultraproduct of equal characteristic discrete valuation rings. So, we can transfer results from equal characteristic, the known case, to results in mixed characteristic. However, the fact that properties only hold with probability one in an ultraproduct accounts for the asymptotic nature of our results. In §10.3, I propose a variant method, using cataproducts instead. Here the asymptotic nature can also be expressed in terms of the *ramification index*, that is to say, the order of the residual characteristic, rather than just the residual characteristic itself. Although this gives often more general results, in terms of more natural invariants, some of the homological problems still elude treatment. We conclude with a result, Theorem 10.3.7, showing how these asymptotic results could nonetheless lead to a positive solution of the corresponding full conjecture, provided we understand the growth rate of these uniform bounds better.

This book also includes two appendices, which contain some applications of the present theory, but also some material used at various points in the main text. Appendix A gives a new construction for the Henselization of a Noetherian local ring. The constructive nature of the process allows us then to define a proto-grading on this Henselization, called the *etale proto-grade*, and apply the theory from Chapter 9 to the ring of algebraic power series rings. Appendix B discusses Boolean rings and some of their generalizations (J-rings, n-Boolean and ω-Boolean rings, periodic rings). In particular, we prove, by means of ultraproducts, some representation theorems analogous to Stone's theorem for Boolean rings, which seem to have been unnoticed hitherto.

Notations and Conventions We follow the common convention to let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively, the natural numbers, the integers, the ring of p-adic integers, the field of rational, of p-adic, of real, and of complex numbers. The q-element field, for q a power of a prime number p, will be denoted $\mathbb{F}_q$; its algebraic closure is denoted $\mathbb{F}_p^{\mathrm{alg}}$. The complement of a set $D \subset W$ is denoted $-D$, and more generally, the difference between two subsets $D, E \subseteq W$ is denoted $D - E$.

All rings are assumed to be commutative. More often than not, the image of an element $a \in A$ under a ring homomorphism $A \to B$ is still denoted a. In particular, IB denotes the ideal generated by the images of elements in the ideal $I \subseteq A$, and $J \cap A$ denotes the ideal of all elements in A whose image lies in the ideal $J \subseteq B$.

Chapter 2
Ultraproducts and Łoś' Theorem

In this chapter, W denotes an infinite set, always used as an index set, on which we fix a non-principal ultrafilter.[1] Given any collection of (first-order) structures indexed by W, we can define their ultraproduct. However, in this book, we will be mainly concerned with the construction of an ultraproduct of rings, an *ultraring* for short, which is then defined as a certain residue ring of their Cartesian product. From this point of view, the construction is purely algebraic, although it is originally a model-theoretic one (we only provide some supplementary background on the model-theoretic perspective). We review some basic properties (deeper theorems will be proved in the later chapters), the most important of which is Łoś' Theorem, relating properties of the approximations with their ultraproduct. When applied to algebraically closed fields, we arrive at a result that is pivotal in most of our applications: the Lefschetz Principle (Theorem 2.4.3), allowing us to transfer many properties between positive and zero characteristic.

2.1 Ultraproducts

We start with the classical definition of ultraproducts via ultrafilters; for different approaches, see §§2.5 and 2.6 below.

2.1.1 *Ultrafilters*

By a *(non-principal) ultrafilter* $\mathfrak{W}$ on W, we mean a collection of infinite subsets of W closed under finite intersection, with the property that for any subset $D \subseteq W$, either D or its complement $-D$ belongs to $\mathfrak{W}$. In particular, the empty set does not belong to $\mathfrak{W}$, and if $D \in \mathfrak{W}$ and E is an arbitrary set containing D, then also

[1] We will drop the adjective 'non-principal' since these are the only ultrafilters we are interested in; if we want to talk about principal ones, we just say *principal filter*; and if we want to talk about both, we say *maximal filter*.

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$E \in \mathfrak{W}$, for otherwise $-E \in \mathfrak{W}$, whence $\emptyset = D \cap -E \in \mathfrak{W}$, contradiction. Since every set in $\mathfrak{W}$ must be infinite, it follows that any co-finite set belongs to $\mathfrak{W}$. The existence of ultrafilters follows from the Axiom of Choice, and we make this set-theoretic assumption henceforth. It follows that for any infinite subset of W, we can find an ultrafilter containing this set.

More generally, a proper collection of subsets of W is called a *filter* if it is closed under intersection and supersets. Any ultrafilter is a filter which is maximal with respect to inclusion. If we drop the requirement that all sets in $\mathfrak{W}$ must infinite, then some singleton must belong to $\mathfrak{W}$; such a filter is called *principal*, and these are the only other maximal filters. A maximal filter is an ultrafilter if and only if it contains the *Frechet filter* consisting of all co-finite subsets (for all these properties, see for instance [81, §4] or [57, §6.4]).

In the remainder of these notes, unless stated otherwise, we fix an ultrafilter $\mathfrak{W}$ on W, and (almost always) omit reference to this fixed ultrafilter from our notation. No extra property of the ultrafilter is assumed, with the one exception described in Remark 8.1.5, which is nowhere used in the rest of our work anyway. Ultrafilters play the role of a decision procedure on the collection of subsets of W by declaring some subsets 'large' (those belonging to $\mathfrak{W}$) and declaring the remaining ones 'small'. More precisely, let o_w be elements indexed by $w \in W$, and let $\mathscr{P}$ be a property. We will use the expressions *almost all* o_w *satisfy property* $\mathscr{P}$ or o_w *satisfies property* $\mathscr{P}$ *for almost all* w as an abbreviation of the statement that there exists a set D in the ultrafilter $\mathfrak{W}$, such that property $\mathscr{P}$ holds for the element o_w, whenever $w \in D$. Note that this is also equivalent with the statement that the set of all $w \in W$ for which o_w has property $\mathscr{P}$, lies in the ultrafilter (read: *is large*).

2.1.2 Ultraproducts

Let O_w be sets, for $w \in W$. We define an equivalence relation on the Cartesian product $O_\infty := \prod O_w$, by calling two sequences (a_w) and (b_w), for $w \in W$, equivalent, if a_w and b_w are equal for almost all w. In other words, if the set of indices $w \in W$ for which $a_w = b_w$ belongs to the ultrafilter. We will denote the equivalence class of a sequence (a_w) by

$$\operatorname*{ulim}_{w \to \infty} a_w, \quad \text{or} \quad \operatorname{ulim} a_w, \quad \text{or} \quad a_\natural.$$

The set of all equivalence classes on $\prod O_w$ is called the *ultraproduct* of the O_w and is denoted

$$\operatorname*{ulim}_{w \to \infty} O_w, \quad \text{or} \quad \operatorname{ulim} O_w, \quad \text{or} \quad O_\natural.$$

If all O_w are equal to the same set O, then we call their ultraproduct the *ultrapower* $O_\natural$ of O. There is a canonical map $O \to O_\natural$, sometimes called the *diagonal embedding*, sending an element o to the image of the constant sequence o in $O_\natural$. To see that it is an injection, assume o' has the same image as o in $O_\natural$. This means that for almost all w, and hence for at least one, the elements o and o' are equal.

Note that the element-wise and set-wise notations are reconciled by the fact that

$$\operatorname*{ulim}_{w\to\infty}\{o_w\} = \{\operatorname*{ulim}_{w\to\infty} o_w\}.$$

The more common notation for an ultraproduct one usually finds in the literature is O^*; in the past, I also have used O_∞, which in this book is reserved to denote Cartesian products. The reason for using the particular notation $O_\natural$ in these notes is because we will also introduce the remaining chromatic products $O_\flat$ and $O_\sharp$ (at least for certain local rings; see Chapters 9 and 8 respectively).

We will also often use the following terminology: if o is an element in an ultraproduct $O_\natural$, then any choice of elements $o_w \in O_w$ with ultraproduct equal to o will be called an *approximation* of o. Although an approximation is not uniquely determined by the element, any two agree almost everywhere. Below we will extend our usage of the term approximation to include other objects as well.

2.1.3 Properties of Ultraproducts

For the following properties, the easy proofs of which are left as an exercise, let O_w be sets with ultraproduct $O_\natural$.

2.1.1 *If Q_w is a subset of O_w for each w, then* $\operatorname{ulim} Q_w$ *is a subset of $O_\natural$.*

In fact, $\operatorname{ulim} Q_w$ consists of all elements of the form $\operatorname{ulim} o_w$, with almost all o_w in Q_w.

2.1.2 *If each O_w is the graph of a function $f_w\colon A_w \to B_w$, then $O_\natural$ is the graph of a function $A_\natural \to B_\natural$, where $A_\natural$ and $B_\natural$ are the respective ultraproducts of A_w and B_w. We will denote this function by*

$$\operatorname*{ulim}_{w\to\infty} f_w \quad \textit{or} \quad f_\natural.$$

Moreover, we have an equality

$$\operatorname*{ulim}_{w\to\infty}(f_w(a_w)) = (\operatorname*{ulim}_{w\to\infty} f_w)(\operatorname*{ulim}_{w\to\infty} a_w), \tag{2.1}$$

for $a_w \in A_w$.

2.1.3 *If each O_w comes with an operation $*_w\colon O_w \times O_w \to O_w$, then*

$$*_\natural := \operatorname*{ulim}_{w\to\infty} *_w$$

is an operation on $O_\natural$. If all (or, almost all) O_w are groups with multiplication $*_w$ and unit element 1_w, then $O_\natural$ is a group with multiplication $*_\natural$ and unit element $1_\natural :=$ $\operatorname{ulim} 1_w$. *If almost all O_w are Abelian groups, then so is $O_\natural$.*