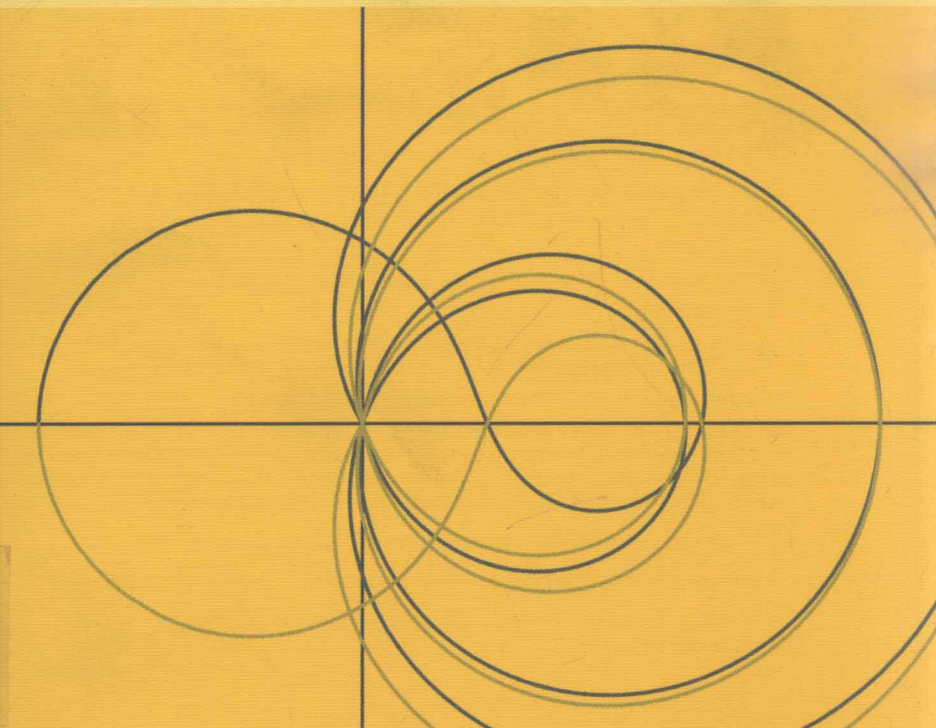
Jörn Steuding

Lecture Notes in Mathematics

Value-Distribution of L-Functions

1877



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Value-Distribution of *L*-Functions

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Preface

L-functions are important objects in modern number theory. They are generating functions formed out of local data associated with either an arithmetic object or with an automorphic form. They can be attached to smooth projective varieties defined over number fields, to irreducible (complex or p-adic) representations of the Galois group of a number field, to a cusp form or to an irreducible cuspidal automorphic representation. All the L-functions have in common that they can be described by an Euler product, i.e., a product taken over prime numbers. In view of the unique prime factorization of integers L-functions also have a Dirichlet series representation. The famous Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

may be regarded as the prototype. L-functions encode in their value-distribution information on the underlying arithmetic or algebraic structure that is often not obtainable by elementary or algebraic methods. For instance, Dirichlet's class number formula gives information on the deviation from unique prime factorization in the ring of integers of quadratic number fields by the values of certain Dirichlet L-functions $L(s,\chi)$ at $s = 1$. In particular, the distribution of zeros of L-functions is of special interest with respect to many problems in multiplicative number theory. A first example is the Riemann hypothesis on the non-vanishing of the Riemann zeta-function in the right half of the critical strip and its impact on the distribution of prime numbers. Another example are L-functions $L(s,E)$ attached to elliptic curves E defined over $\mathbb{Q}$. The yet unproved conjecture of Birch and Swinnerton-Dyer claims that $L(s,E)$ has a zero at $s = 1$ whose order is equal to the rank of the Mordell-Weil group of the elliptic curve E.

These notes present recent results in the value-distribution theory of such L-functions with an emphasis on the phenomenon of universality. The starting point of this theory is Bohr's achievement at the first half of the twentieth century. He proved denseness results and first limit theorems for the values of the Riemann zeta-function. Maybe the most remarkable result concerning the value-distribution of $\zeta(s)$ is Voronin's universality theorem from 1975, which roughly states that any non-vanishing analytic function can be approximated uniformly by certain shifts of the zeta-function in the critical strip. More precisely: *let* $0 < r < \frac{1}{4}$ *and suppose that* $g(s)$ *is a non-vanishing continuous function on the disc* $|s| \leq r$ *which is analytic in its interior. Then, for any* $\epsilon > 0$, *there exists a real number* τ *such that*

$$\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+\mathrm{i}\tau\right)-g(s)\right|<\epsilon;$$

moreover, the set of these τ *has positive lower density:*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\left\{\tau\in[0,T]:\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+\mathrm{i}\tau\right)-g(s)\right|<\epsilon\right\}>0.$$

This is a remarkable property! We say that $\zeta(s)$ is *universal* since it allows uniform approximation of a large class of functions. Voronin's universality theorem, in a spectacular way, indicates that Riemann's zeta-function is a transcendental function; clearly, rational functions cannot be universal. In some literature the validity of the Riemann hypothesis for abelian varieties (proved by Hasse for elliptic curves and by Weil in the general case) is regarded as evidence for the truth of Riemann's hypothesis for $\zeta(s)$. However, the zeta-function of an abelian variety is a rational function and so its value-distribution is of a rather different type.

The Linnik–Ibragimov conjecture asserts that any Dirichlet series (which has a *sufficiently rich* value-distribution) is universal. Meanwhile we know quite many universal Dirichlet series; for instance, Dirichlet L-functions (Voronin, 1975), Dedekind zeta-functions (Reich, 1980), Lerch zeta-functions (Laurinčikas, 1997), and L-functions associated with newforms (Laurinčikas, Matsumoto and Steuding, 2003). One aim of these notes is to prove an extension of Voronin's universality theorem for a large class of L-functions which covers (at least conjecturally) all known L-functions of number-theoretical significance.

These notes are organized as follows. In the introduction, we give an overview on the value-distribution theory of the classical Riemann zeta-function and Dirichlet L-functions; also we touch some allied zeta-functions which we will not consider in detail in the following chapters. In Chap. 2. we introduce a class $\tilde{\mathcal{S}}$ of Dirichlet series, satisfying certain analytic and arithmetic axioms. The members of this class are the main actors in the sequel. Roughly speaking, an L-function in $\tilde{\mathcal{S}}$ has a polynomial Euler product and satisfies some hypothesis which may be regarded as some kind of prime number theorem; besides, we require analytic continuation to the left of the half-plane of absolute convergence for the associated Dirichlet series in addition with some growth condition. The axioms defining $\tilde{\mathcal{S}}$ are kept quite general and therefore they may appear to be rather abstract and technical; however, as we shall discuss later for many examples (in Chaps. 6, 12 and 13), they hold (or at least they are expected to hold) for all L-functions of number theoretical interest. This abstract setting has the advantage that we can derive a rather general universality theorem.

Our proof of universality, in the main part, relies on Bagchi's probabilistic approach from 1981. For the sake of completeness we briefly present in Chap. 3 some basic facts from probability theory and measure theory. In Chap. 4, we prove along the lines of Laurinčikas' extension of Bagchi's method a limit theorem (in the sense of weakly convergent probability measures) for functions in the class $\tilde{\mathcal{S}}$. In the following chapter we give the proof of the main result, a universality theorem for L-functions in $\tilde{\mathcal{S}}$. The proof depends on the limit theorem of the previous chapter and the so-called positive density method, recently introduced by Laurinčikas and Matsumoto to tackle L-functions attached to cusp forms. Furthermore, we discuss the phenomenon of discrete universality; here the attribute *discrete* means that the shifts τ are taken from arithmetic progressions. This concept of universality was introduced by Reich in 1980.

In Chap. 6, we introduce the Selberg class $\mathcal{S}$ consisting of Dirichlet series with Euler product and a functional equation of Riemann-type (and a bit more). It is a folklore conjecture that the Selberg class consists of all automorphic L-functions. We study basic facts about $\mathcal{S}$ and discuss the main conjectures, in particular, the far-reaching Selberg conjectures on primitive elements. We shall see that the class $\tilde{\mathcal{S}}$ fits rather well into the setting of the Selberg class $\mathcal{S}$ (especially with respect to Selberg's conjectures). Hence, our general universality theorem extends to the Selberg class, unconditionally for many of the classical L-function and conditionally to all elements of $\mathcal{S}$ subject to some widely believed but rather deep conjectures. However, the Selberg class is too small with respect to universality; for instance, a Dirichlet L-function to an imprimitive character does not lie in the Selberg class (by lack of an appropriate functional equation) but it is known to be universal. Furthermore, some important L-functions are only conjectured to lie in the Selberg class, and, in spite of this, for some of them we can derive universality unconditionally.

In the following chapter, we consider the value-distribution of Dirichlet series $\mathcal{L}(s)$ with functional equation in the complex plane. Following Levinson's approach from the 1970s, we shall prove asymptotic formulae for the c-values of $\mathcal{L}$, i.e., roots of the equation $\mathcal{L}(s) = c$, and give applications in Nevanlinna theory. In particular, we give an alternative proof of the Riemann-von Mangoldt formula for the elements in the Selberg class.

The main themes of Chap. 8 are almost periodicity and the Riemann hypothesis. Universality has an interesting feedback to classical problems. Bohr observed that the Riemann hypothesis for Dirichlet L-functions associated with non-principal characters is equivalent to almost periodicity in the right half of the critical strip. Applying Voronin's universality theorem, Bagchi was able to extend this result to the zeta-function in proving that if the Riemann zeta-function can approximate itself uniformly in the sense of Voronin's theorem, then Riemann's hypothesis is true, and vice versa. We sketch an extension of Bagchi's theorem to other L-functions.

Chapter 9 deals with the problem of effectivity. The known proofs of universality are ineffective, giving neither bounds for the first approximating shift τ nor for their density (with the exception of particular results due to Garunkštis, Good, and Laurinčikas). We give explicit upper bounds for the density of universality; more precisely, we prove upper bounds for the frequency with which a certain class of target functions (analytic isomorphisms) can be uniformly approximated. Moreover, we apply effective results from the theory of inhomogeneous diophantine approximation to prove several explicit estimates for the value-distribution in the half-plane of absolute convergence.

In Chap. 10, we discuss further applications of universality, most of them classical, e.g., an extension of Bohr's and Voronin's results concerning the value-distribution inside the critical strip, and the functional independence which covers Ostrowski's solution of the Hilbert problem on the hyper-transcendence of the zeta-function and some of its generalizations. Here a function is called hyper-transcendental, if it does not satisfy any algebraic differential equation. Further, we study the value-distribution of linear combinations of (strongly) universal Dirichlet series. A subtle consequence of this *strong* concept of universality, and a big contrast to L-functions, can be found in the distribution of zeros off the critical line. Very likely a (universal) Dirichlet series satisfying a functional equation of Riemann-type has either *many* zeros to the right of the critical line (as a generic Dirichlet series with periodic coefficients) or *none* (as it is expected for L-functions). This seems to be the heart of many secrets in the value-distribution theory of Dirichlet series.

Chapter 11 deals with Dirichlet series associated with periodic arithmetical functions. In general, these functions do not have an Euler product but they are additively related to Dirichlet L-functions. Consequently, they share certain properties with L-functions, e.g., a functional equation similar to the one for Riemann's zeta-function. We prove universality for a large class of these Dirichlet series; in contrast to L-functions they can approximate uniformly analytic functions having zeros (provided their Dirichlet coefficients are not multiplicative). Moreover, we study joint universality for Hurwitz zeta-functions with rational parameters.

We conclude with joint universality; here *joint* stands for simultaneous uniform approximation. In Chap. 12, we prove a theorem which reduces joint universality for L-functions in $\tilde{\mathcal{S}}$ to a denseness property in a related function space. Of course, we cannot have joint universality for any set of L-functions: for example, $\zeta(s)$ and $\zeta(s)^2$ cannot approximate any given pair of admissible target functions simultaneously. However, we shall prove that in some instances twists of $\mathcal{L} \in \tilde{\mathcal{S}}$ with pairwise non-equivalent characters fulfill this condition (e.g., Dirichlet L functions). In the following chapter we present several further applications. For instance, we prove joint universality for Artin

L-functions (which lie in the Selberg class if and only if the deep Artin conjecture is true). This universality theorem holds unconditionally despite the fact that Artin L-functions might have infinitely many poles in their strip of universality; this was first proved by Bauer in 2003 by a tricky argument.

At the end of these notes an appendix on the history of the general phenomenon of universality in analysis is given. It is known that universality is a quite regularly appearing phenomenon in limit processes, but among all these universal objects only universal Dirichlet series are explicitly known. At the end an index and a list of the notations and axioms which were used are given.

Value-distribution theory for L-functions with emphasis on aspects of universality was treated in the monographs of Karatsuba and Voronin [166] and Laurinčikas [186]. However, after the publication of these books, many new results and applications were discovered; we refer the reader to the surveys of Laurinčikas [196] and of Matsumoto [242] for some of the progress made in the meantime. The content of this book forms an extract of the authors habilitation thesis written at Frankfurt University in 2003. We have added Chaps. 12 and 13 on joint universality and its applications as well as several remarks and comments concerning the progress obtained in the meantime. Unfortunately, we could not include the most current contributions as, for example, the promising work [245] of Mauclaire which relates universality with almost periodicity.

I am very grateful to Springer for publishing these notes; especially, I want to thank Stefanie Zöller and Catriona M. Byrne from Springer, the editors of the series Lecture Notes in Mathematics, and, of course, the anonymous referees for their excellent work, their valuable remarks and corrections. Furthermore, I am grateful to my family, my friends and my colleagues for their interest and support, in particular those from the Mathematics Departments at the universities of Frankfurt, Madrid, and Würzburg. Especially, I would like to thank Ramūnas Garunkštis and Antanas Laurinčikas for introducing me to questions concerning universality, Ernesto Girondo, Aleksander Ivić, Roma Kačinskaitė, Kohji Matsumoto, Georg Johann Rieger, Jürgen Sander, Wolfgang Schwarz, and Jürgen Wolfart for the fruitful discussions, helpful remarks and their encouragement. Last but not least, I would like to thank my wife Rasa.

Jörn Steuding
Würzburg, December 2006

Contents

1

Introduction

The grandmother of all zeta-functions is the Riemann zeta-function.

David Ruelle

In this introduction we give some hints for the importance of the Riemann zeta-function for analytic number theory and present first classic results on its amazing value-distribution due to Harald Bohr but also the remarkable universality theorem of Voronin (including a sketch of his proof). Moreover, we introduce Dirichlet L-functions and other generalizations of the zeta-function, discuss their relevance in number theory and comment on their value-distribution. For historical details we refer to Narkiewicz's monograph [277] and Schwarz's surveys [317, 318].

1.1 The Riemann Zeta-Function and the Distribution of Prime Numbers

The Riemann zeta-function is a function of a complex variable $s = \sigma + it$, for $\sigma > 1$ given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}; \tag{1.1}$$

here and in the sequel the letter p always denotes a prime number and the product is taken over all primes. The Dirichlet series, and the Euler product, converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. The identity between the Dirichlet series and the Euler product was discovered by Euler [76] in 1737 and can be regarded as an analytic version of the unique prime factorization of integers. The Euler product gives a first glance on the intimate connection between the zeta-function and the distribution of prime numbers. A first immediate consequence is Euler's proof of the infinitude of the primes. Assuming that there were only finitely many primes, the product in (1.1) is finite, and therefore convergent for $s = 1$, contradicting the fact that the Dirichlet series defining $\zeta(s)$ reduces to the divergent harmonic series as $s \to 1+$. Hence, there exist infinitely many prime numbers. This fact is well known since Euclid's elementary proof, but the analytic access gives deeper knowledge on the distribution of the prime numbers. It was the young Gauss [94] who conjectured in 1791 for the number $\pi(x)$ of primes $p \le x$ the asymptotic formula

$$\pi(x) \sim \mathrm{li}(x), \tag{1.2}$$

where the logarithmic integral is given by

$$\mathrm{li}(x) = \lim_{\epsilon \to 0+} \left\{ \int_0^{1-\epsilon} + \int_{1+\epsilon}^{x} \right\} \frac{du}{\log u} = \int_2^x \frac{du}{\log u} - 1.04\ldots;$$

this integral is a principal value in the sense of Cauchy. Gauss' conjecture states that, in first approximation, the number of primes $\le x$ is asymptotically $\frac{x}{\log x}$. By elementary means, Chebyshev [54, 55] proved around 1850 that for sufficiently large x

$$0.921\ldots \le \pi(x) \frac{\log x}{x} \le 1.055\ldots.$$

Furthermore, he showed that if the limit

$$\lim_{x \to \infty} \pi(x) \frac{\log x}{x}$$

exists, the limit is equal to one, which supports relation (1.2).

Riemann was the first to investigate the Riemann zeta-function as a function of a complex variable. In his only one but outstanding paper [310] on number theory from 1859 he outlined how Gauss' conjecture could be proved by using the function $\zeta(s)$. However, at Riemann's time the theory of functions was not developed sufficiently far, but the open questions concerning the zeta-function pushed the research in this field quickly forward. We shall briefly discuss Riemann's memoir. First of all, by partial summation

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s \int_N^{\infty} \frac{[u] - u}{u^{s+1}} \, du; \tag{1.3}$$

here and in the sequel $[u]$ denotes the maximal integer less than or equal to u. This gives an analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ except for a simple pole at $s = 1$ with residue 1. This process can be continued to the left half-plane and shows that $\zeta(s)$ is analytic throughout the whole complex plane except for $s = 1$. Riemann gave the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \tag{1.4}$$

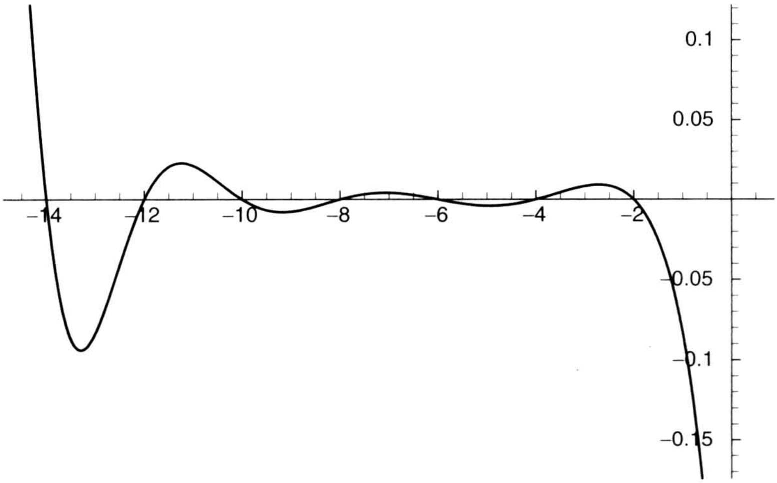

Fig. 1.1. $\zeta(s)$ in the range $s \in [-14.5, 0.5]$

where $\Gamma(s)$ denotes Euler's Gamma-function; it should be noted that Euler [76] had partial results in this direction (namely, for integral s and for half-integral s; see [7]). In view of the Euler product (1.1) it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. It follows from the functional equation and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2n$ with $n \in \mathbb{N}$ (see Fig. 1.1 for the *first* trivial zeros). All other zeros of $\zeta(s)$ are said to be non-trivial, and we denote them by $\varrho = \beta + i\gamma$. Obviously, they have to lie inside the so-called critical strip $0 \leq \sigma \leq 1$, and it is easily seen that they are non-real. The functional equation (1.4), in addition with the identity

$$\zeta(\overline{s}) = \overline{\zeta(s)},$$

shows some symmetries of $\zeta(s)$. In particular, the non-trivial zeros of $\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\sigma = \frac{1}{2}$. It was Riemann's ingenious contribution to number theory to point out how the distribution of these non-trivial zeros is linked to the distribution of prime numbers. Riemann conjectured that the number $N(T)$ of non-trivial zeros $\varrho = \beta + i\gamma$ with $0 < \gamma \leq T$ (counted according multiplicities) satisfies the asymptotic formula

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e}.$$

This was proved in 1895 by von Mangoldt [235, 236] who found more precisely

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{1.5}$$

Riemann worked with the function $t \mapsto \zeta(\frac{1}{2} + it)$ and wrote that very likely all roots t are real, i.e., all non-trivial zeros lie on the so-called critical line

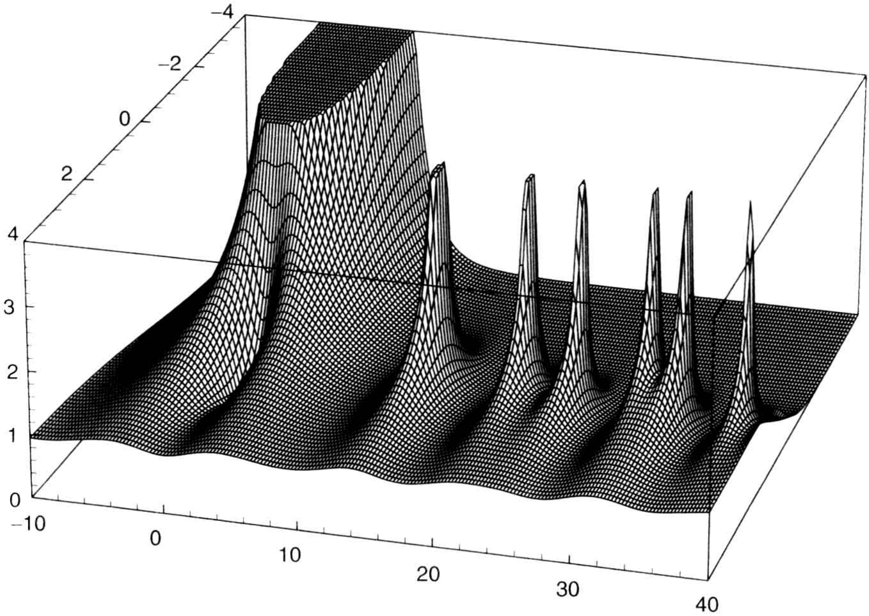

Fig. 1.2. The reciprocal of the absolute value of $\zeta(s)$ for $\sigma \in [-4, 4], t \in [-10, 40]$. The zeros of $\zeta(s)$ appear as poles

$\sigma = \frac{1}{2}$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann's Hypothesis. *$\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$.*

In support of his conjecture, Riemann calculated some zeros; the first one with positive imaginary part is $\varrho = \frac{1}{2} + i14.134\ldots$ (see Fig. 1.2 and also Fig. 8.1).* Furthermore, Riemann conjectured that there exist constants A and B such that

$$\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \exp(A + Bs)\prod_{\varrho}\left(1 - \frac{s}{\varrho}\right)\exp\left(\frac{s}{\varrho}\right).$$

This was shown by Hadamard [113] in 1893 (recall the Hadamard product theorem from the theory of functions). Finally, Riemann conjectured the so-called explicit formula which states that

* In 1932, Siegel [329] published an account of Riemann's work on the zeta-function found in Riemann's private papers in the archive of the university library in Göttingen. It became evident that behind Riemann's speculation there was extensive analysis and computation.

$$\pi(x) + \sum_{n=2}^{\infty} \frac{\pi(x^{1/n})}{n} = \mathrm{li}(x) - \sum_{\substack{\varrho=\beta+\mathrm{i}\gamma \\ \gamma>0}} \left(\mathrm{li}(x^{\varrho}) + \mathrm{li}(x^{1-\varrho})\right) \tag{1.6}$$

$$+ \int_x^{\infty} \frac{\mathrm{d}u}{u(u^2-1)\log u} - \log 2$$

for any $x \geq 2$ not being a prime power (otherwise a term $\frac{1}{2k}$ has to be added on the left-hand side, where $x = p^k$); the appearing integral logarithm is defined by

$$\mathrm{li}(x^{\beta+\mathrm{i}\gamma}) = \int_{(-\infty+\mathrm{i}\gamma)\log x}^{(\beta+\mathrm{i}\gamma)\log x} \frac{\exp(z)}{z+\delta\mathrm{i}\gamma}\,\mathrm{d}z,$$

where $\delta = +1$ if $\gamma > 0$ and $\delta = -1$ otherwise. The explicit formula was proved by von Mangoldt [235] in 1895 as a consequence of both product representations for $\zeta(s)$, the Euler product (1.1) on the one hand and the Hadamard product on the other hand.

Riemann's ideas led to the first proof of Gauss' conjecture (1.2), the celebrated prime number theorem, by Hadamard [114] and de la Vallée-Poussin [357] (independently) in 1896. We give a very brief sketch (for the details we refer to Ivić [141]). For technical reasons it is of advantage to work with the logarithmic derivative of $\zeta(s)$ which is for $\sigma > 1$ given by

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where the von Mangoldt Λ-function is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

A lot of information concerning the prime counting function $\pi(x)$ can be recovered from information about

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \log p + \mathrm{O}\left(x^{1/2} \log x\right).$$

Partial summation gives

$$\pi(x) \sim \frac{\psi(x)}{\log x}.$$

First of all, we shall express $\psi(x)$ in terms of the zeta-function. If c is a positive constant, then

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^s}{s}\,\mathrm{d}s = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } 0 < x < 1. \end{cases} \tag{1.8}$$

This yields the Perron formula: for $x \notin \mathbb{Z}$ and $c > 1$,

$$\psi(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, ds. \tag{1.9}$$

Moving the path of integration to the left, we find that the latter expression is equal to the corresponding sum of residues, that are the residues of the integrand at the pole of $\zeta(s)$ at $s = 1$, at the zeros of $\zeta(s)$, and at the pole of the integrand at $s = 0$. The main term turns out to be

$$\mathrm{Res}_{\,s=1} \left\{ -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \right\} = \lim_{s \to 1} (s-1) \left(\frac{1}{s-1} + \mathrm{O}(1) \right) \frac{x^s}{s} = x,$$

whereas each non-trivial zero ϱ gives the contribution

$$\mathrm{Res}_{\,s=\varrho} \left\{ -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \right\} = -\frac{x^\varrho}{\varrho}.$$

By the same reasoning, the trivial zeros contribute

$$\sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} = \frac{1}{2} \log \left(1 - \frac{1}{x^2} \right).$$

Incorporating the residue at $s = 0$, this leads to the the *exact* explicit formula

$$\psi(x) = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{1}{2} \log \left(1 - \frac{1}{x^2} \right) - \log(2\pi),$$

which is equivalent to Riemann's formula (1.6). Notice that the right-hand side of this formula is not absolutely convergent. If $\zeta(s)$ would have only finitely many non-trivial zeros, the right-hand side would be a continuous function of x, contradicting the jumps of $\psi(x)$ for prime powers x. However, going on it is much more convenient to cut the integral in (1.9) at $t = \pm T$ which leads to the truncated version

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\varrho}{\varrho} + \mathrm{O} \left(\frac{x}{T} (\log(xT))^2 \right), \tag{1.10}$$

valid for all values of x. Next we need information on the distribution of the non-trivial zeros. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov [359] and Korobov [173] (independently) who proved

$$\zeta(s) \neq 0 \qquad \text{in} \quad \sigma \geq 1 - \frac{c}{(\log|t| + 3)^{1/3} (\log\log(|t| + 3))^{2/3}},$$

where c is some positive absolute constant; the first complete proof due to Richert appeared in Walfisz [366]. In addition with the Riemann–von Mangoldt formula (1.5) one can estimate the sum over the non-trivial zeros in (1.10). Balancing out T and x, we obtain the prime number theorem with the strongest existing remainder term: