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Regularity Theory for Quasilinear Elliptic Systems and Monge - Ampère Equations in Two Dimensions

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Dedicated to Professor E. Heinz

PREFACE

These lecture notes have been written as an introduction to the characteristic theory for two-dimensional Monge-Ampère equations, a theory largely developed by H. Lewy and E. Heinz which has never been presented in book form. An exposition of the Heinz-Lewy theory requires auxiliary material which can essentially be found in various monographs, but which is presented here, in part because the focus is different, and also because these notes have an introductory character and are intended to be essentially self contained: Included are excerpts from the regularity theory for elliptic systems, the theory of pseudoanalytic functions and the theory of conformal mappings. References are usually scarce in the text; the sources are listed in the introduction.

These notes grew out of a seminar given in the Department of Mathematics of the University of Kentucky in Lexington during the Fall of 1988. I gratefully acknowledge the support of N.S.F. grant RII-8610671 and the Commonwealth of Kentucky through the Kentucky EPSCoR Program. It is a pleasure to thank Ronald Gariepy for inviting me to give the seminar, and I wish to express my sincere gratitude to all participants for their warm hospitality. Special thanks also to David Adams, Craig Evans and Neil Trudinger, whose interest encouraged me to actually write these notes. I am grateful for the corrections of George Paulik, and I appreciate the patience and dedication of Julie Hill in typing the bulk of the manuscript.

Iowa City, May 1990

INTRODUCTION

An outline of the book, some historical remarks and the sources are given in this introductory section.

The present notes are mainly concerned with the "characteristic" theory for elliptic Monge–Ampère equations. This theory was largely developed by H. Lewy [L 3, 4] and E. Heinz [H 1–3, 5–7, 11, 12], motivated and based on the characteristic theory for hyperbolic equations as presented in Courant–Hilbert [CH] and Hadamard [HA 2] and on the "characteristic" theory for hyperbolic surfaces as developed by Darboux [DB] and F. Rellich [RE 2], who made the connection between the characteristic theory for differential equations and the surface theory.

Most noteworthy seems to be Appendix 4 to Chapter V of [CH] concerning the special role of the hyperbolic Monge–Ampère equation in the characteristic theory, namely the fact that the characteristic system of this fully nonlinear equation consists of only five equations instead of eight, a property that it shares with quasilinear equations. This fact was instrumental when Lewy founded the "characteristic" theory for elliptic Monge–Ampère equations.

Remarkable, and nowadays standard knowledge, are Appendices 1 and 2 to Chapter V of [CH] on Lewy's method [L 2] to accomplish the change from the hyperbolic to the elliptic case via a complex substitution.

In the "characteristic" theory for hyperbolic surfaces, asymptotic line parameters are constructed. A complex substitution (Rellich [R 2]) yields the "elliptic" case of convex surfaces, in which the "characteristic" variables are the conjugate isothermal parameters.

To be more precise, when introducing conjugate isothermal parameters x, y for a locally convex surface Σ, the second fundamental form

$$\text{II}_\Sigma = L\,du^2 + 2M\,du\,dv + N\,dv^2 \tag{1}$$

is reduced to

$$\text{II}_\Sigma = \Lambda\,(dx^2 + dy^2), \qquad \Lambda > 0. \tag{2}$$

The mapping $(u(x,y), v(x,y))$ satisfies a second order elliptic system, which can be written in the form

$$Lu = \Gamma^1_{11}\,|Du|^2 + \Gamma^1_{12}\,Du\cdot Dv + \Gamma^1_{22}\,|Du|^2, \tag{3}$$

$$Lv = \Gamma^2_{11}\,|Du|^2 + \Gamma^2_{12}\,Du\cdot Dv + \Gamma^2_{22}\,|Du|^2, \tag{4}$$

$$L = -\frac{1}{\sqrt{K}}\left[\frac{\partial}{\partial x}\left[\sqrt{K}\frac{\partial}{\partial x}\right] + \frac{\partial}{\partial y}\left[\sqrt{K}\frac{\partial}{\partial y}\right]\right]. \tag{5}$$

Here K is the Gauß curvature and the Γ^k_{ij}'s are the Christoffel symbols of the first fundamental form

$$I_\Sigma = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{6}$$

The fact that the coefficients of the system (3,4) depend only on the coefficients of the first fundamental form (6) and their derivatives is truly remarkable, a theorema egregium in the sense of Gauß.

The "Darboux system" (3,4) is the harmonic map system if K is constant. It can be derived at least formally by a complex substitution from the classical Darboux system for hyperbolic surfaces [DB].

The main thrust of the present notes is to present the Heinz–Lewy "characteristic" theory for elliptic Monge–Ampère equations. Consider the characteristic form

$$ds^2 = (r+C)\,dx^2 + 2(s-B)\,dx\,dy + (t+A)\,dy^2 \tag{7}$$

$$= a\,dx^2 + 2b\,dx\,dy + c\,dy^2 \tag{8}$$

associated with the elliptic Monge–Ampère equation

$$Ar + 2Bs + Ct + (rt - s^2) = E, \tag{9}$$

or equivalently

$$(r+C)(t+A) - (s-B)^2 = \Delta, \tag{10}$$

$$\Delta = AC - B^2 + E > 0, \tag{11}$$

for a given solution $z = z(x,y)$. Here $p = z_x$, $q = z_y$, $r = z_{yy}$, $s = z_{xy}$, $t = z_{yy}$. New variables u, v are introduced such that

$$ds^2 = \Lambda(du^2 + dv^2), \qquad \Lambda \neq 0. \tag{12}$$

We shall call the parameters u, v "characteristic", although not in the literal sense. Thus a conformal map with respect to the Riemannian metric ds^2 is constructed and the corresponding Beltrami system is

$$x_u = \frac{b x_v + c y_v}{\sqrt{\Delta}}, \tag{13}$$

$$x_v = \frac{-b x_u - c y_u}{\sqrt{\Delta}}. \tag{14}$$

As far as the Monge–Ampère equation is concerned, one wishes to obtain information about the second derivatives, i.e., about the coefficients of the characteristic form. This means that the coefficients of the Beltrami system are "unknown". The difficulty is resolved by the observation (a theorema egregium of sorts) that the inverse mapping $(x(u,v), y(u,v))$, in addition to solving a Beltrami system, satisfies a quasilinear elliptic system of second order with quadratic growth in the gradient of the solution mapping of the form

$$Lx = h_1 |Dx|^2 + h_2 Dx \cdot Dy + h_3 |Dy|^2 + h_4 Dx \wedge Dy, \tag{15}$$

$$Ly = \tilde{h}_1 |Dx|^2 + \tilde{h}_2 Dx \cdot Dy + \tilde{h}_3 |Dy|^2 + \tilde{h}_4 Dx \wedge Dy, \tag{16}$$

$$L = -\frac{1}{\sqrt{\Delta}} \left[\frac{\partial}{\partial u}\left[\sqrt{\Delta}\frac{\partial}{\partial u}\right] + \frac{\partial}{\partial v}\left[\sqrt{\Delta}\frac{\partial}{\partial v}\right]\right]. \tag{17}$$

The coefficients $h_1, \dots, \tilde{h}_4$ can be computed in terms of p, q and certain derivatives of A, B, C and Δ. The "characteristic" system (15, 16) reduces to

$$Lx = 0, \tag{18}$$

$$Ly = 0 \tag{19}$$

in the case of the simple Monge–Ampère equation

$$rt - s^2 = \Delta > 0. \tag{20}$$

This means that one should study diffeomorphic solutions of quasilinear elliptic second order systems. The corresponding conformality relations can be written in the form

$$\frac{t+A}{\sqrt{\Delta}} = \frac{|Dx|^2}{J(x,y)}, \tag{21}$$

$$-\frac{s-B}{\sqrt{\Delta}} = \frac{Dx \cdot Dy}{J(x,y)}, \tag{22}$$

$$\frac{r+C}{\sqrt{\Delta}} = \frac{|Dy|^2}{J(x,y)}, \tag{23}$$

$$J(x,y) = x_u y_v - x_v y_u, \tag{24}$$

and they can be used as a "dictionary", namely to translate the information obtained for the second order system, such as regularity or a priori estimates, into information for the Monge–Ampère equation.

It turns out that the regularity theory for Monge–Ampère equations can be presented much more directly by studying the Legendre–like variable transformation

$$u = x, \tag{25}$$

$$v = q \tag{26}$$

for the simple Monge–Ampère equation (20). This is of course motivated by the characteristic theory. The "characteristic" system (18, 19) is replaced by the equation

$$y_{uu} + (\Delta y_v)_v = 0. \tag{27}$$

This and the extension to general Monge–Ampère equations (9) are the contents of CHAPTER 3. The Campanato regularity technique is developed for Monge–Ampère equations, thus providing a scheme to prove sharp a priori estimates assuming the knowledge of bounds for the absolute values of the solution and its derivatives up to the second order. In order to continue paraphrasing Lewy's remarks from 1934 ([L3] loc. cit.) it seems, however, that the estimation of the second derivatives themselves requires much deeper insight. Purely local second derivative estimates can, at this point in time, only be shown via the characteristic theory which is the topic of Chapter 9. The Schauder technique could have been employed to yield the a priori estimates mentioned above (see Schulz [SZ3]), but, it seems, not the regularity results. The presentation of Chapter 3 is based on Schulz [SZ1–4] and Schulz–Liao [SL]. Historical references for the Legendre–like transformation (25, 26) are Darboux [DB], Heinz [H5], Hartman–Wintner [HW1,3], Jörgens [JÖ1].

Chapter 3 requires the regularity theory for linear elliptic equations, in particular the Campanato technique. This is presented in CHAPTER 2 based on Campanato [C3,4] and Giaquinta [GI].

Basic tools, in particular the concept of Hölder continuity, which are needed in Chapter 2 and later are presented in CHAPTER 1. The sources are: Campanato [C1,2], Evans–Gariepy [EG], Giaquinta [GI], Gilbarg–Trudinger [GT] and Heinz [H2].

Quasilinear elliptic second order systems are studied in CHAPTERS 2, 5 and 8. In Chapter 2 (Section 4), the regularity theory for univalent solutions is presented based on Schulz [SZ5]. In Chapters 5 and 8 diffeomorphic solutions of Heinz–Lewy type systems (15, 16) are studied. The non–vanishing of the Jacobian is shown together with an a priori estimate from below. Chapter 5 deals with a special case which can be proved with the similarity principle. The reference is Heinz [H2]. Chapter 8 is about the general case without the similarity principle. The presentation is based on Heinz [H6, 11] and Schulz [SZ5]. The general case via the similarity principle is not presented here. This topic would be an extension of Heinz [H12]. The fundamental historical reference is Lewy [L4], whose ideas are incorporated in the text, in particular in the proof of Proposition 8.1.2. Other references are Berg [BG] and Bers [BS3], who studied univalent solutions of linear systems.

Function theoretic tools which are needed in Chapter 5 are presented in CHAPTER 4. The main theorem is the similarity principle for pseudoanalytic functions by Bers and Vekua [BS 1, 2], [VE 1, 2], a $\bar{\partial}$–proof of which is presented, and a Harnack type inequality. The sources are: Bers [BS 2], Goursat [GO], Heinz [H 2] and Vekua [VE 2].

Tools needed in Chapter 8 are presented in CHAPTER 7. Function theoretic properties of elliptic equations are presented which cannot directly be derived from the similarity principle. The local behavior of functions satisfying elliptic differential inequalities is studied. The sources are: Hartman–Wintner [HW 2], Heinz [H 6] and Schulz [SZ 5].

Conformal mappings with respect to a Riemannian metric are studied in CHAPTER 6. The focus here is however somewhat different than in the standard literature about the Riemann mapping theorem. Our interest lies in the connections between uniformization and second order elliptic systems. The sources are: Ahlfors [AF], Heinz [H 3], Schiffer–Spencer [SHS], Schulz [SZ 6] and Vekua [VE 2]. Some results are used in Chapter 7, but the major applications are presented in Chapters 9 and 10, namely the connection between Monge–Ampère equations and quasilinear elliptic systems and the role of the Darboux system in the theory of convex surfaces.

CHAPTER 9 can be considered the core of the current notes. It is concerned with the characteristic theory for elliptic Monge–Ampère equations as outlined at the beginning of this introduction. Characteristic parameters are introduced by employing Chapter 6. The results of Chapters 2, 5, 8 on second order elliptic systems are translated into results for Monge–Ampère equations via the conformality relations (21, 22, 23). A priori estimates are thus derived for the absolute values of the second derivatives r, s, t of solutions of Monge–Ampère equations. The presentation is based on Heinz [H 3, 7] and Schulz [SZ 6].

Some geometric applications of Chapter 9 are discussed in CHAPTER 10, such as convex graphs of prescribed Gauß curvature and convex surfaces via the Darboux equation (a Monge–Ampère type equation which is different from the Darboux system). The main thrust of this chapter however is the investigation of locally convex surfaces without utilizing Monge–Ampère equations. Conjugate isothermal parameters x, y are introduced, and the Darboux system (3, 4) is derived as the "characteristic" system. A priori estimates are derived for the coefficients of the second fundamental form (1). This is based on Heinz [H 6] and Schulz [SZ 7].

Many interesting topics could not be covered in these lecture notes, most notably the special case of harmonic mappings, Jörgens's theorem for the equation $rt - s^2 = 1$ and applications to minimal surfaces. The investigation of the differential inequality $\alpha \leq rt - s^2 \leq \beta$ would have high priority for inclusion in an expanded version of these notes.

In addition to listing the sources at the end of the notes, the BIBLIOGRAPHY includes some related work, in particular two–dimensional Monge–Ampère equations and classical surface theory in three–space. The focus here is two–dimensional and work on multidimensional problems was not included.

The notation used is usually explained as it occurs. Because of the variety of the topics presented, we experimented with various notation such as complex, two–dimensional and multidimensional index notation. For most of the more subtle sections, it seemed necessary to employ the two–dimensional notation used in this introduction. There is a NOTATION INDEX on page 121, basic notation is explained on the following page.

BASIC NOTATION

$B_R = B_R(x)$ is the open ball or disc in $\mathbb{R}^n$ of radius R centered at x $(n \geq 2)$.
$\mathcal{N} = \mathcal{N}(x)$ is an open neighborhood of x, i.e., an open set containing x.
Ω denotes an open subset of $\mathbb{R}^n$; Ω is a domain if it is also connected.
$\Omega' \subset\subset \Omega$ means that the closure of Ω' is compact and contained in Ω.
e_i is the i^{th} standart unit vector in $\mathbb{R}^n$.
$|x| = (\Sigma x_i^2)^{1/2}$ for a point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$.
$|\alpha| = \alpha_1 + \ldots + \alpha_n$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{Z}^n$, $\alpha_i \geq 0$.
$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$.
$D_\alpha u = \partial u/\partial x_\alpha$, $D_{\alpha\beta} u = \partial^2 u/\partial x_\alpha \partial x_\beta, \ldots$ $(\alpha, \beta, \ldots = 1, \ldots, n)$.
$Du = (D_1 u, \ldots, D_n u)$ is the gradient of u.
$D^\alpha u = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$.
$C^k_{loc}(\Omega)$ is the set of functions having continuous derivatives of order $\leq k$ in Ω $(0 \leq k \leq \infty)$.
$C^k(\Omega)$ is the set of functions in $C^k_{loc}(\Omega)$ with finite C^k-norm

$$\|u\|_{C^k(\Omega)} = \sum_{\ell=1}^{k} |D^\ell u| = \sum_{\ell=1}^{k} \sup_{|\alpha|=\ell} \sup_{\Omega} |D^\alpha u|.$$

$C^k_0(\Omega)$ is the set of functions u in $C^k_{loc}(\Omega)$ with compact support in Ω, i.e., the closure of the set on which $u \neq 0$ is compact (and contained in Ω).
$C^{k,1}_{loc}(\Omega)$ is the set of functions in $C^k_{loc}(\Omega)$ whose k^{th} order derivatives are Lipschitz continuous in every $\Omega' \subset\subset \Omega$.
$L^p(\Omega)$ is the set of equivalence classes of measurable functions u on Ω which agree a.e. with finite L^p-norm

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p dx\right]^{1/p}.$$

Note: In practice we usually do not identify two L^p-functions which agree a.e. and work instead with the precise representative which is given by the Lebesgue differentiation theorem.
$L^p_{loc}(\Omega)$ is the set of functions in $L^p(\Omega')$ for every $\Omega' \subset\subset \Omega$.
$\sup_\Omega u$ denotes the essential supremum of u, i.e. the quantity $\operatorname{ess\,sup} u = \|u^+\|_{L^\infty(\Omega)}$.
$\operatorname{osc}_\Omega u$ is the essential oscillation of u in Ω, i.e. equal to $\operatorname{ess\,osc} u = \sup_\Omega u - \inf_\Omega u$.
$C = C(\ldots)$ denotes a constant which depends only on the quantities that are listed in parentheses. The letter C will denote various constants which may change from line to line. We choose constants to be ≥ 1, if possible. Generic constants that are < 1 are usually denoted by the lower case letter c.

TABLE OF CONTENTS

Chapter 1. INTEGRAL CRITERIA FOR HÖLDER CONTINUITY

1.1. Sobolev functions

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with a typical point denoted by $x = (x_1, \dots, x_n)$. The Sobolev space $W^{k,p}(\Omega)$, $k = 0, 1, 2, \dots$, $1 \leq p \leq \infty$, is the space of functions $u \in L^p(\Omega)$ with weak partial derivatives up to order k in $L^p(\Omega)$. Here, for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$, $v = D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ is the α^{th} weak derivative of u in Ω if

$$(1.1) \qquad \int_\Omega v\, \eta\, dx = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \eta\, dx$$

for all $\eta \in C_0^\infty(\Omega)$.

The function u belongs to $W^{k,p}_{loc}(\Omega)$ if $u \in W^{k,p}(\Omega')$ for each $\Omega' \subset\subset \Omega$. Vector valued Sobolev functions $u = (u^1, \dots, u^N) \in W^{k,p}(\Omega, \mathbb{R}^N)$ are defined in the obvious way.

$W^{k,p}(\Omega)$ is a Banach space if equipped with the Sobolev norm

$$(1.2) \qquad \|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} = \left[\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p\, dx\right]^{1/p}$$

for $1 \leq p < \infty$, and, with $\sup$ denoting the essential supremum,

$$(1.3) \qquad \|u\|_{k,\infty} = \|u\|_{W^{k,\infty}(\Omega)} = \sup_\Omega \sum_{|\alpha| \leq k} |D^\alpha u| .$$

$W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$.

Sobolev functions can be approximated by smooth functions on Ω: If $1 \leq p < \infty$, then a theorem of Meyers–Serrin [MS] states for arbitrary Ω that

$$(1.4) \qquad W^{k,p}(\Omega) = H^{k,p}(\Omega),$$

which is the completion of $\{u \in C^k_{loc}(\Omega) \mid \|u\|_{k,p} < \infty\}$ with respect to the norm $\|\cdot\|_{k,p}$. In the case $p = \infty$,

$$(1.5) \qquad W^{k,\infty}_{loc}(\Omega) = C^{k-1,1}_{loc}(\Omega),$$

the space of functions whose $(k-1)^{st}$ derivatives are locally Lipschitz continuous in Ω.

A Sobolev function of class $W^{1,p}_{loc}(\mathbb{R}^n)$ is precisely represented by a function u which is locally absolutely continuous on almost all lines, i.e., $u_i(t) = u(\dots, x_{i-1}, t, x_i, \dots)$ is absolutely continuous on compact subintervals of $\mathbb{R}$ for almost every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and $Du_i \in L^p_{loc}(\mathbb{R}^n)$.

1.2. The Dirichlet growth theorem

Let $B_R = B_R(x)$ be the open ball of radius R in $\mathbb{R}^n$ centered at x, and let

(1.6) $$u_{x,R} = ⨍_{B_R} u(y)\,dy = \frac{1}{|B_R|}\int_{B_R(x)} u(y)\,dy\,.$$

Definition 1.2.1. Let μ be a constant, $0 < \mu < 1$. A function $u:\Omega \longrightarrow \mathbb{R}$ is *Hölder continuous* with exponent μ in Ω, if the quantity (Hölder seminorm)

(1.7) $$[u]_\mu^\Omega = \sup_{\substack{x',x''\in\Omega \\ x'\neq x''}} \frac{|u(x')-u(x'')|}{|x'-x''|^\mu}$$

is finite. u is locally Hölder continuous in Ω, if u is Hölder continuous in every Ω', $\Omega' \subset\subset \Omega$. $C^{k,\mu}(\Omega)$ $(C^{k,\mu}_{loc}(\Omega))$ is the set of functions $u \in C^k(\Omega)$ $(C^k_{loc}(\Omega))$ whose k^{th} order derivatives are (locally) Hölder continuous with exponent μ in Ω $(k = 0,1,2,\dots)$. The $C^{k,\mu}$-norm of u is

(1.8) $$\|u\|_{C^{k,\mu}(\Omega)} = \|u\|_{C^k(\Omega)} + [D^k u]_\mu^\Omega = \|u\|_{C^k(\Omega)} + \sup_{|\alpha|=k}[D^\alpha u]_\mu^\Omega\,.$$

The following "Dirichlet growth theorem" by Morrey (see [MO 2]) guarantees Hölder continuity of certain Sobolev functions:

Theorem 1.2.2. *Let* $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$. *Suppose that for some constants* μ, M, $0 < \mu \le 1$, $M > 0$,

(1.9) $$\int_{B_R} |Du|^p\,dy \le M^p R^{n-p+p\mu}$$

for all balls B_R *in* Ω. *Then* $u \in C^{0,\mu}_{loc}(\Omega)$, *and in each ball* B_R *with* $B_{3R} \subset \Omega$, *the estimate*

(1.10) $$\operatorname*{osc}_{B_R} u \le C M R^\mu$$

holds with a constant C *which depends only on* n, p, μ.

Proof: By approximation, we may assume that $u \in C^1_{loc}(\Omega)$. Let $B_{3R}(x_0) \subset \Omega$ and let $x' \in B_R(x_0)$. Then $B_R(x_0) \subset B_{2R}(x') \subset \Omega$, and

$$|u(x')-u(x_0)| \le |u(x')-u_{x_0,R}| + |u(x_0)-u_{x_0,R}|$$
$$\le 2^n ⨍_{B_{2R}(x')} |u(y)-u(x')|\,dy + ⨍_{B_R(x_0)} |u(y)-u(x_0)|\,dy\,.$$

To estimate an integral of the form

$$\int_{B_R(x)} |u(y)-u(x)|\,dy\,,$$

note that

$$|u(y)-u(x)| \le \int_0^1 |Du(x+t\,(y-x))|\,dt\;|y-x|\,. \tag{1.11}$$

By integrating with respect to y over $B_R(x)$,

$$\begin{aligned}
\int_{B_R(x)} |u(y)-u(x)|\,dy &\le \int_0^1\int_{B_R(x)} |Du(x+t\,(y-x))|\;|y-x|\,dy\,dt \\
&= \int_0^1 t^{-n-1}\int_{B_{tR}(x)} |Du(\xi)|\;|\xi-x|\,d\xi\,dt \\
&\le \int_0^1 t^{-n-1}\Big[\int_{B_{tR}} |Du|^p\,d\xi\Big]^{1/p}\Big[\int_{B_{tR}} |\xi-x|^{p/(p-1)}\,d\xi\Big]^{(p-1)/p} dt \\
&\le C\,M\int_0^1 t^{-n-1}\,(tR)^{(n-p+p\mu)/p}\,(tR)^{(n+p/(p-1))(p-1)/p}\,dt \\
&= C\,M\,R^{n+\mu}\int_0^1 t^{\mu-1}\,dt \\
&\le C\,M\,R^{n+\mu},
\end{aligned}$$

incorporating the assumed "Dirichlet growth." The theorem follows. □

1.3. Poincaré inequalities

Lemma 1.3.1. *Assume that* $u \in C^1(B_R)$, $B_R = B_R(x_0)$. *Then the inequality*

$$\int_{B_R} |u(y)-u(x)|^p\,dy \le C\,R^{n+p-1}\int_{B_R} |Du(y)|^p\,|y-x|^{1-n}\,dy \tag{1.12}$$

holds for all $x \in B_R$ *with a constant* C *which depends only on* n, p.

Proof: Recall (1.11), which implies, by Hölder's inequality, that

$$|u(y)-u(x)|^p \le \int_0^1 |Du(x+t\,(y-x))|^p\,dt\;|y-x|^p\,.$$

By integrating with respect to y over $\partial B_r(x)\cap B_R$, $0<r<2R$,