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Hamiltonian Dynamics Theory and Applications

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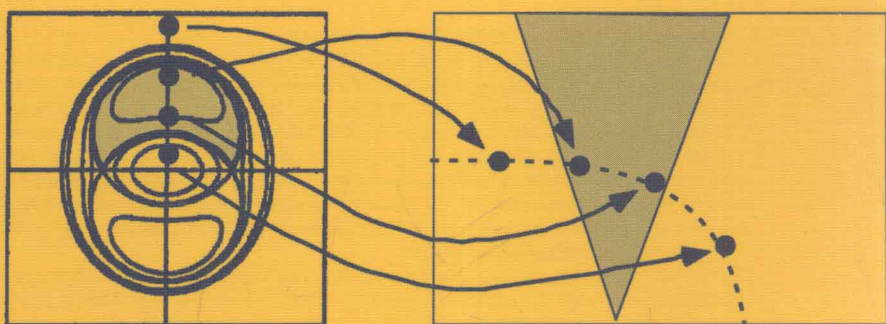

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Giancarlo Benettin
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Hamiltonian Dynamics Theory and Applications

Lectures given at the
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Preface

" *Nous sommes donc conduit à nous proposer le problème suivant: Étudier les équations canoniques*

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}\ , \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}$$

en supposant que la function F peut se développer suivant les puissances d'un paramètre très petit μ de la manière suivante:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots\ ,$$

en supposant de plus que F_0 ne dépend que des x et est indépendent des y; et que F_1, $F_2, \dots$ sont des fonctions périodiques de période 2π par rapport aux y. "

This is all of the contents of §13 in the first volume of the celebrated treatise *Les méthodes nouvelles de la mécanique céleste* of Poincaré, published in 1892.

In more usual notations and words, the problem is to investigate the dynamics of a canonical system of differential equations with Hamiltonian

$$(1) \qquad H(p,q,\varepsilon) = H_0(p) + \varepsilon H_1(p,q) + \varepsilon^2 H_2(p,q) + \dots\ ,$$

where $p \equiv (p_1,\dots,p_n) \in \mathcal{G} \subset \mathbf{R}^n$ are action variables in the open set $\mathcal{G}$, $q \equiv (q_1,\dots,q_n) \in \mathbf{T}^n$ are angle variables, and ε is a small parameter.

The lectures by Giancarlo Benettin, Jacques Henrard and Sergej Kuksin published in the present book address some of the many questions that are hidden behind the simple sentence above.

1. A Classical Problem

It is well known that the investigations of Poincaré were motivated by a classical problem: the stability of the Solar System. The three volumes of the

Méthodes Nouvelles had been preceded by the memoir *Sur le problème des trois corps et les équations de la dynamique; mémoire couronné du prix de S. M. le Roi Oscar II le 21 janvier 1889.*

It may be interesting to recall the subject of the investigation, as stated in the announcement of the competition for King Oscar's prize:

> " *A system being given of a number whatever of particles attracting one another mutually according to Newton's law, it is proposed, on the assumption that there never takes place an impact of two particles to expand the coordinates of each particle in a series proceeding according to some known functions of time and converging uniformly for any space of time.* "

In the announcement it is also mentioned that the question was suggested by a claim made by Lejeune–Dirichlet in a letter to a friend that he had been able to demonstrate the stability of the solar system by integrating the differential equations of Mechanics. However, Dirichlet died shortly after, and no reference to his method was actually found in his notes.

As a matter of fact, in his memoir and in the *Méthodes Nouvelles* Poincaré seems to end up with different conclusions. Just to mention a few results of his work, let me recall the theorem on generic non–existence of first integrals, the recurrence theorem, the divergence of classical perturbation series as a typical fact, the discovery of asymptotic solutions and the existence of homoclinic points.

Needless to say, the work of Poincaré represents the starting point of most of the research on dynamical systems in the XX–th century. It has also been said that the memoir on the problem of three bodies is "the first textbook in the qualitative theory of dynamical systems", perhaps forgetting that the qualitative study of dynamics had been undertaken by Poincaré in a *Mémoire sur les courbes définies par une équation différentielle*, published in 1882.

2. KAM Theory

Let me recall a few known facts about the system (1). For $\varepsilon = 0$ the Hamiltonian possesses n first integrals $p_1, \ldots, p_n$ that are independent, and the orbits lie on invariant tori carrying periodic or quasi–periodic motions with frequencies $\omega_1(p), \ldots, \omega_n(p)$, where $\omega_j(p) = \frac{\partial H_0}{\partial p_j}$. This is the unperturbed dynamics. For $\varepsilon \neq 0$ this plain behaviour is destroyed, and the problem is to understand how the dynamics actually changes.

The classical methods of perturbation theory, as started by Lagrange and Laplace, may be resumed by saying that one tries to prove that for $\varepsilon \neq 0$ the system (1) is still integrable. However, this program encountered major difficulties due to the appearance in the expansions of the so called *secular* *terms*, generated by resonances among the frequencies. Thus the problem become that of writing solutions valid for all times, possibly expanded in power series of the parameter ε. By the way, the role played by resonances is indeed at the basis of the non–integrability in classical sense of the perturbed system, as stated by Poincaré.

A relevant step in removing secular terms was made by Lindstedt in 1882. The underlying idea of Lindstedt's method is to look for *a single solution which is characterized by fixed frequencies*, $\lambda_1, \ldots, \lambda_n$ say, and which is close to the unperturbed torus with the same frequencies. This allowed him to produce series expansions free from secular terms, but he did not solve the problem of the presence of small denominators, i.e., denominators of the form $\langle k, \lambda\rangle$ where $0 \neq k \in \mathbf{Z}^n$. Even assuming that these quantities do not vanish (i.e., excluding resonances) they may become arbitrarily small, thus making the convergence of the series questionable.

In tome II, chap. XIII, § 148–149 of the *Méthodes Nouvelles* Poincaré devoted several pages to the discussion of the convergence of the series of Lindstedt. However, the arguments of Poincaré did not allow him to reach a definite conclusion:

> " ... *les séries ne pourraient–elles pas, par example, converger quand ... le rapport n_1/n_2 soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport n_1/n_2 est assujetti à une autre condition analogue à celle que je viens d'énoncer un peu au hasard)?*
>
> *Les raisonnements de ce chapitre ne me permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable.* "

Here, n_1, n_2 are the frequencies, that we have denoted by λ_1, λ_2.

The problem of the convergence was settled in an indirect way 60 years later by Kolmogorov, when he announced his celebrated theorem. In brief, *if the perturbation is small enough, then most (in measure theoretic sense) of the unperturbed solutions survive, being only slightly deformed.* The surviving invariant tori are characterized by some strong non–resonance conditions, that in Kolmogorov's note was identified with the so called *diophantine condition*, namely $\bigl|\langle k, \lambda\rangle\bigr| \geq \gamma |k|^{-\tau}$ for some $\gamma > 0$, $\tau > n-1$ and for all non–zero $k \in \mathbf{Z}^n$. This includes the case of the frequencies chosen "un peu au hasard" by Poincaré. It is often said that Kolmogorov announced his theorem without publishing the proof; as a matter of fact, his short communication contains a sketch of the proof where all critical elements are clearly pointed out. Detailed proofs were published later by Moser (1962) and Arnold (1963); the theorem become thus known as KAM theorem.

The argument of Kolmogorov constitutes only an indirect proof of the convergence of the series of Lindstedt; this has been pointed out by Moser in 1967. For, the proof invented by Kolmogorov is based on an infinite sequence of canonical transformations that give the Hamiltonian the appropriate normal form

$$H(p,q) = \langle \lambda, p \rangle + R(p,q) \ ,$$

where $R(p,q)$ is at least quadratic in the action variables p. Such a Hamiltonian possesses the invariant torus $p = 0$ carrying quasi–periodic motions with frequencies λ. This implies that the series of Lindstedt must converge, since they give precisely the form of the solution lying on the invariant torus. However, Moser failed to obtain a *direct* proof based, e.g., on Cauchy's classical method of majorants applied to Lindstedt's expansions in powers of ε. As discovered by Eliasson, this is due to the presence in Lindstedt's classical series of terms that grow too fast, due precisely to the small denominators, but are cancelled out by internal compensations (this was written in a report of 1988, but was published only in 1996). Explicit constructive algorithms taking compensations into account have been recently produced by Gallavotti, Chierchia, Falcolini, Gentile and Mastropietro.

In recent years, the perturbation methods for Hamiltonian systems, and in particular the KAM theory, has been extended to the case of PDE's equations. The lectures of Kuksin included in this volume constitute a plain and complete presentation of these recent theories.

3. Adiabatic Invariants

The theory of adiabatic invariants is related to the study of the dynamics of systems with slowly varying parameters. That is, the Hamiltonian $H(q,p\,;\lambda)$ depends on a parameter $\lambda = \varepsilon t$, with ε small. The typical simple example is a pendulum the length of which is subjected to a very slow change – e.g., a periodic change with a period much longer than the proper period of the pendulum. The main concern is the search for quantities that remain close to constants during the evolution of the system, at least for reasonably long time intervals. This is a classical problem that has received much attention at the beginning of the the XX–th century, when the quantities to be considered were identified with the actions of the system.

The usefulness of the action variables has been particularly emphasized in the book of Max Born *The Mechanics of the Atom*, published in 1927. In that book the use of action variables in quantum theory is widely discussed. However, it should be remarked that most of the book is actually devoted to Hamiltonian dynamics and perturbation methods. In this connection it may be interesting to quote the first few sentences of the preface to the german edition of the book:

> " *The title "Atomic Mechanics" given to these lectures . . . was chosen to correspond to the designation "Celestial Mechanics". As the latter term covers that branch of theoretical astronomy which deals*

with with the calculation of the orbits of celestial bodies according to mechanical laws, so the phrase "Atomic Mechanics" is chosen to signify that the facts of atomic physics are to be treated here with special reference to the underlying mechanical principles; an attempt is made, in other words, at a deductive treatment of atomic theory. "

The theory of adiabatic invariants is discussed in this volume in the lectures of J. Henrard. The discussion includes in particular some recent developments that deal not just with the slow evolution of the actions, but also with the changes induced on them when the orbit crosses some critical regions. Making reference to the model of the pendulum, a typical case is the crossing of the separatrix. Among the interesting phenomena investigated with this method one will find, e.g., the capture of the orbit in a resonant regions and the sweeping of resonances in the Solar System.

4. Long–Time Stability and Nekhoroshev's Theory

Although the theorem of Kolmogorov has been often indicated as the solution of the problem of stability of the Solar System, during the last 50 years it became more and more evident that it is not so. An immediate remark is that the theorem assures the persistence of a set of invariant tori with relative measure tending to one when the perturbation parameter ε goes to zero, but the complement of the invariant tori is open and dense, thus making the actual application of the theorem to a physical system doubtful, due to the indeterminacy of the initial conditions. Only the case of a system of two degrees of freedom can be dealt with this way, since the invariant tori create separated gaps on the invariant surface of constant energy. Moreover, the threshold for the applicability of the theorem, i.e., the actual value of ε below which the theorem applies, could be unrealistic, unless one considers very localized situations. Although there are no general definite proofs in this sense, many numerical calculations made independently by, e.g., A. Milani, J. Wisdom and J. Laskar, show that at least the motion of the minor planets looks far from being a quasi–periodic one.

Thus, the problem of stability requires further investigation. In this respect, a way out may be found by proving that some relevant quantities, e.g., the actions of the system, remain close to their initial value for a long time; this could lead to a sort of "effective stability" that may be enough for physical application. In more precise terms, one could look for an estimate $\bigl|p(t) - p(0)\bigr| = O(\varepsilon^a)$ for all times $|t| < T(\varepsilon)$, were a is some number in the interval $(0, 1)$ (e.g., $a = 1/2$ or $a = 1/n$), and $T(\varepsilon)$ is a "large" time, in some sense to be made precise.

The request above may be meaningful if we take into consideration some characteristics of the dynamical system that is (more or less accurately) described by our equations. In this case the quest for a "large" time should be interpreted as *large with respect to some characteristic time of the physical system, or comparable with the lifetime of it*. For instance, for the nowadays accelerators a characteristic time is the period of revolution of a particle of the beam and the typical lifetime of the beam during an experiment may be a few days, which may correspond to some 10^{10} revolutions; for the solar system the lifetime is the estimated age of the universe, which corresponds to some 10^{10} revolutions of Jupiter; for a galaxy, we should consider that the stars may perform a few hundred revolutions during a time as long as the age of the universe, which means that a galaxy does not really need to be much stable in order to exist.

From a mathematical viewpoint the word "large" is more difficult to explain, since there is no typical lifetime associated to a differential equation. Hence, in order to give the word "stability" a meaning in the sense above it is essential to consider the dependence of the time T on ε. In this respect the continuity with respect to initial data does not help too much. For instance, if we consider the trivial example of the equilibrium point of the differential equation $\dot{x} = x$ one will immediately see that if $x(0) = x_0 > 0$ is the initial point, then we have $x(t) > 2x_0$ for $t > T = \ln 2$ no matter how small is x_0; hence T may hardly be considered to be "large", since it remains constant as x_0 decreases to 0. Conversely, if for a particular system we could prove, e.g., that $T(\varepsilon) = O(1/\varepsilon)$ then our result would perhaps be meaningful; this is indeed the typical goal of the theory of adiabatic invariants.

Stronger forms of stability may be found by proving, e.g., that $T(\varepsilon) \sim 1/\varepsilon^r$ for some $r > 1$; this is indeed the theory of complete stability due to Birkhoff. As a matter of fact, the methods of perturbation theory allow us to prove more: in the inequality above one may actually choose r depending on ε, and increasing when $\varepsilon \to 0$. In this case one obtains the so called *exponential stability*, stating that $T(\varepsilon) \sim \exp(1/\varepsilon^b)$ for some b. Such a strong result was first stated by Moser (1955) and Littlewood (1959) in particular cases. A complete theory in this direction was developed by Nekhoroshev, and published in 1978.

The lectures of Benettin in this volume deal with the application of the theory of Nekhoroshev to some interesting physical systems, including the collision of molecules, the classical problem of the rigid body and the triangular Lagrangian equilibria of the problem of three bodies.

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The success of the school has been assured by the high level of the lectures and by the enthusiasm of the participants. A particular thankfulness is due

to Giancarlo Benettin, Jacques Henrard and Sergej Kuksin, who accepted not only to profess their excellent lectures, but also to contribute with their writings to the preparation of this volume

Milano, March 2004

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Contents

Physical Applications of Nekhoroshev Theorem and Exponential Estimates

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1 Introduction

The purpose of these lectures is to discuss some physical applications of Hamiltonian perturbation theory. Just to enter the subject, let us consider the usual situation of a nearly-integrable Hamiltonian system,

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi)\ , \qquad \begin{aligned} I &= (I_1,\ldots,I_n) \in \mathcal{B} \subset \mathbb{R}^n \\ \varphi &= (\varphi_1,\ldots,\varphi_n) \in \mathbb{T}^n\ , \end{aligned} \tag{1.1}$$

$\mathcal{B}$ being a ball in $\mathbb{R}^n$. As we shall see, such a framework is often poor and not really adequate for some important physical applications, nevertheless it is a natural starting point. For $\varepsilon = 0$ the phase space is decomposed into invariant tori $\{I\} \times \mathbb{T}^n$, see figure 1, on which the flow is linear:

$$I(t) = I^o\ , \qquad \varphi(t) = \varphi^o + \omega(I^o)t\ ,$$

with $\omega = \frac{\partial h}{\partial I}$. For $\varepsilon \neq 0$ one is instead confronted with the nontrivial equations

$$\dot I = -\varepsilon \frac{\partial f}{\partial \varphi}(I,\varphi)\ , \qquad \dot\varphi = \omega(I) + \varepsilon \frac{\partial f}{\partial I}(I,\varphi)\ . \tag{1.2}$$

Different stategies can be used in front of such equations, all of them sharing the elementary idea of "averaging out" in some way the term $\frac{\partial f}{\partial \varphi}$, to show that, in convenient assumptions, the evolution of the actions (if any) is very slow. In perturbation theory, "slow" means in general that $\|I(t) - I(0)\|$ remains small, for small ε, at least for $t \sim 1/\varepsilon$ (that is: the evolution is slower than the trivial *a priori* estimate following (1.2)). Throughout these lectures, however,

* Gruppo Nazionale di Fisica Matematica and Istituto Nazionale di Fisica della Materia

"slow" will have the stronger meaning of "exponentially slow", namely (with reference to any norm in $\mathbb{R}^n$)

$$\|I(t) - I(0)\| < \mathcal{I}\,(\varepsilon/\varepsilon_*)^b \qquad \text{for} \qquad |t| < \mathcal{T} e^{(\varepsilon_*/\varepsilon)^a} \ , \qquad (1.3)$$

$\mathcal{T}, \mathcal{I}, a, b, \varepsilon_*$ being positive constants. It is worthwhile to mention that stability results for times long, though not infinite, are very welcome in physics: indeed every physical observation or experiment, and in fact every physical model (like a frictionless model of the Solar System) are sensible only on an appropriate time scale, which is possibly long but is hardly infinite.[2] Results of perpetual stability are certainly more appealing, but the price to be paid — like ignoring a dense open set in the phase space, as in KAM theory — can be too high, in view of a clear physical interpretation.

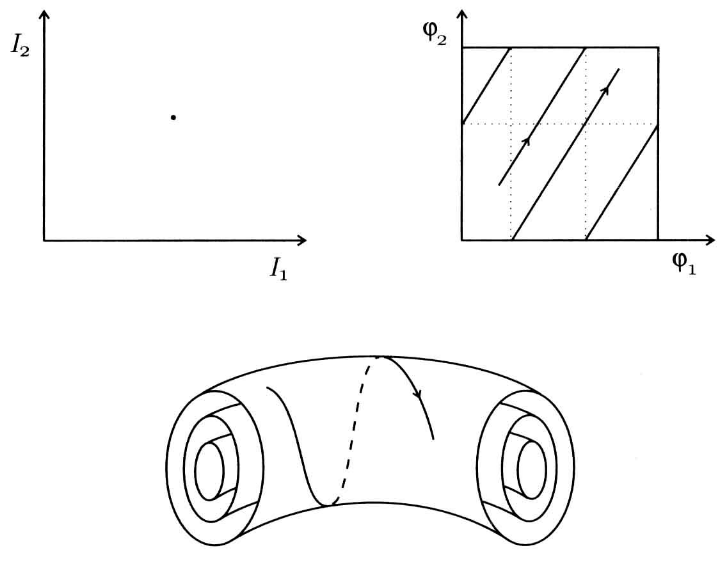

Fig. 1. *Quasi periodic motion on invariant tori.*

Poincaré, at the beginning of his *Mèthodes Nouvelles de la Méchanique Céleste* [Po1], stressed with emphasis the importance of systems of the form (1.1), using for them the strong expression *"Problème général de la dynamique"*. As a matter of fact, systems of the form (1.1), or natural generalizations of them, are met throughout physics, from Molecular Physics to Celestial Mechanics. Our choice of applications — certainly non exhausting — will be the following:

[2] Littlewood in '59 produced a stability result for long times, $t \sim \exp(\log \varepsilon)^2$, in connection with the triangular Lagrangian points, and his comment was: "this is not eternity, but is a considerable slice of it" [Li].

- Boltzmann's problem of the specific heats of gases: namely understanding why some degrees of freedom, like the fast internal vibration of diatomic molecules, are essentially decoupled ("frozen", in the later language of quantum mechanics), and do not appreciably contribute to the specific heats.
- The fast-rotations of the rigid body (equivalently, a rigid body in a weak force field, that is a perturbation of the Euler–Poinsot case). The aim is to understand the conditions for long-time stability of motions, with attention, on the opposite side, to the possible presence of chaotic motions. Some attention is deserved to "gyroscopic phenomena", namely to the properties of motions close to the (unperturbed) stationary rotations.
- The stability of elliptic equilibria, with special emphasis on the "triangular Lagrangian equilibria" L_4 and L_5 in the (spatial) circular restricted three body problem.

There would be other interesting applications of perturbation theory, in different fields: for example problems of magnetic confinement, the numerous stability problems in asteroid belts or in planetary rings, the stability of bounches of particles in accelerators, the problem of the physical realization of ideal constraints. We shall not enter them, nor we shall consider any of the recent extensions to systems with infinitely many degres of freedom (localization of excitations in nonlinear systems; stability of solutions of nonlinear wave equations; selected problems from classical electrodynamics...), which would be very interesting, but go definitely bejond our purposes.

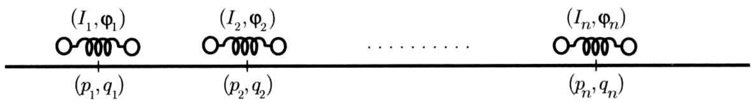

Fig. 2. *An elementary one–dimensional model of a diatomic gas.*

As already remarked, physical systems, including those we shall deal with, typically do not fit the too simple form (1.1), and require a generalization: for example

$$H(I, \varphi, p, q) = h(I) + \varepsilon f(I, \varphi, p, q) \ , \tag{1.4}$$

or also

$$H(I, \varphi, p, q) = h(I) + \mathcal{H}(p, q) + \varepsilon f(I, \varphi, p, q) \ , \tag{1.5}$$

the new variables (q, p) belonging to $\mathbb{R}^{2m}$ (or to an open subset of it, or to a manifold). In problems of molecular dynamics, for the specific heats, the new degrees of freedom represent typically the centers of mass of the molecules (see figure 2), and the Hamiltonian fits the form (1.5). Instead in the rigid body dynamics, as well as in many problems in Celestial Mechanics, p, q are still action–angle variables, but the actions do not enter the unperturbed Hamiltonian, and this makes a relevant difference. The unperturbed Hamiltonian, if it does not depend on all actions, is said to be *properly degenerate*, and the absent actions are themselves called degenerate. For the Kepler problem, the degenerate actions represent the eccentricity and the inclination of the orbit; for the Euler-Poinsot rigid body they determine the orientation in space of the angular momentum. The perturbed Hamiltonian, for such systems, fits (1.4). Understanding the behavior of degenerate variables is physically important, but in general is not easy, and requires assumptions on the perturbation.[3] Such an investigation is among the most interesting ones in perturbation theory.

As a final introductory remark, let us comment the distinction, proposed in the title of these lectures, between "exponential estimates" and "Nekhoroshev theorem".[4] As we shall see, some perturbative problems concern systems with essentially constant frequencies. These include isochronous systems, but also some anisochronous systems for which the frequencies stay nevertheless almost constants during the motion, as is the case of molecular collisions. Such systems require only an analytic study: in the very essence, it is enough to construct a single normal form, with an exponentially small remainder, to prove the desired result. We shall address these problems with the generic expression "exponential estimates". We shall instead deserve the more specific expression "Nekhoroshev theorem", or theory, for problems which are effectively anisochronous, and require in an essential way, to be overcome, suitable geometric assumptions, like convexity or "steepness" of the unperturbed Hamiltonian h (and occasionally assumptions on the perturbation, too). The geometrical aspects are in a sense the heart of Nekhoroshev theorem, and certainly constitute its major novelty. As we shall see, geometry will play an absolutely essential role both in the study of the rigid body and in the case of the Lagrangian equilibria.

These lectures are organized as follows: Section 2 is devoted to exponential estimates, and includes, after a general introduction to standard perturbative methods, some applications to molecular dynamics. It also includes an account of an approximation proposed by Jeans and by Landau and Teller, which looks alternative to standard methods, and seems to work excellently in connection with molecular collisions. Section 3 is fully devoted to the Jeans–Landau–Teller approximation, which is revisited within a mathematically well posed perturbative scheme. Section 4 contains an application of exponential estimates to Statistical Mechanics, namely to the Boltzmann question about the possible existence of long equilibrium times in classical gases. Section 5 contains a general introduction to Nekhoroshev theorem. Section 6 is devoted

[3] This is clear if one considers, in (1.4), a perturbation depending only on (p, q): these variables, for suitable f, can do anything on a time scale $1/\varepsilon$.

[4] Such a distinction is not common in the literature, where the expression "Nekhoroshev theorem" is often ued as a synonymous of stability results for exponentially long times.

to the applications of Nekhoroshev theory to Euler–Poinsot perturbed rigid body, while Section 7 is devoted to the application of the theory to elliptic equilibria, in particular to the stability of the so–called Lagrangian equilibrium points L_4, L_5 in the (spatial) circular restricted three body problem.

The style of the lectures will be occasionally informal; the aim is to provide a general overview, with emphasis when possible on the connections between different applications, but with no possibility of entering details. Proofs will be absent, or occasionally reduced to a sketch when useful to explain the most relevant ideas. (As is well known to researchers active in perturbation theory, complete proofs are long, and necessarily include annoying parts, so for them we forcely demand to the literature.) Besides rigorous results, we shall also produce heuristic results, as well as numerical results; understanding a physical system requires in fact, very often, the cooperation of all of these investigation tools.

Most results reported in these lectures, and all the ideas underlying them, are fruit on one hand of many years of intense collaboration with Luigi Galgani, Antonio Giorgilli and Giovanni Gallavotti, from whom I learned, in the essence, all I know; on the other hand, they are fruit of the intense collaboration, in the last ten years, with my colleagues Francesco Fassò and more recently Massimiliano Guzzo. I wish to express to all of them my gratitude. I also wish to thank the director of CIME, Arrigo Cellina, and the director of the school, Antonio Giorgilli, for their proposal to give these lectures. I finally thank Massimiliano Guzzo for having reviewed the manuscript.

2 Exponential Estimates

We start here with a general result concerning exponential estimates in exactly isochronous systems. Then we pass to applications to molecular dynamics, for systems with either one or two independent frequencies.

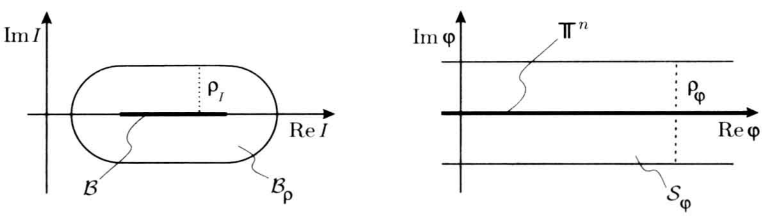

Fig. 3. *The complex extended domains of the action–angle variables.*

A. Isochronous Systems

Let us consider a system of the form (1.1), with linear and thus isochronous h:

$$H(I, \varphi) = \omega \cdot I + \varepsilon f(I, \varphi) \; . \tag{2.1}$$