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Topology, Geometry, and Algebra: Interactions and New Directions

Conference on Algebraic Topology
in Honor of R. James Milgram
August 17–21, 1999
Stanford University

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PREFACE

In August 1999 a meeting was held at Stanford University on the subject of "Topology, Geometry and Algebra: Interactions and New Directions". The goal of the conference was to bring together distinguished researchers from a variety of areas related to algebraic topology and its applications. The list of invited speakers included: Greg Arone, Sylvain Cappell, Jon Carlson, Fred Cohen, Jim Davis, Don Davis, Tom Goodwillie, Yakov Eliashberg, Tom Farrell, Mike Hopkins, Eleny Ionel, Ronnie Lee, Ib Madsen, Mark Mahowald, Bob Oliver, Peter Oszvath, John Rognes, Abigail Thompson, Ulrike Tillmann and Efim Zelmanov.

A number of topics were covered in the lectures, including homotopy theory, moduli spaces, group cohomology, manifold theory, algebraic K-theory, low dimensional topology, symplectic geometry, etc. Special emphasis was made on breadth and interactions between different areas. A large number of postdocs, graduate students and established mathematicians attended the lectures. This volume contains twelve refereed papers on a wide variety of topics reflecting the nature of the conference.

The conference was partly held to celebrate Jim Milgram's sixtieth birthday and to recognize his enormous contributions to algebraic topology over the past 35 years. His powerful mathematics, deep insight and breadth of knowledge have been an inspiration to many of us, and it is a great pleasure for us to dedicate this volume to him. The first paper in this volume is an outline of Jim Milgram's main mathematical contributions.

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On Jim Milgram's Mathematical Work

Gunnar Carlsson

Jim Milgram has been one of the leading figures in algebraic topology over the last 35 years. He has contributed to all parts of the subject, and his work has been characterized by tremendous depth and an unsurpassed calculational power. His influence will continue to be felt throughout algebraic topology for the foreseeable future. Personally, I was one of Jim's earlier Ph.D. students. He was a superb advisor, combining the right amount of help in crucial situations with an insistence on independence. In particular, he communicated his mathematical taste, which includes an emphasis on concrete problems and computations, very effectively. I am very grateful to him for all his work with me during my student days, as well as for his friendship since them. In this note I will attempt to summarize the high points of Jim's work. The description I will give will not be exhaustive, and it is colored by my own biases and interests.

I. Iterated Loop Spaces

In the mid 1950's, I. M. James [J] had constructed the combinatorial model for the loop space of a suspension which carries his name. Many mathematicians (including Kudo-Araki [K-A], Browder [B1], and Dyer-Lashof [D-L]) had extended this work by studying the homology of iterated loop spaces of iterated suspensions. All this work was aimed at homology computations, though, and did not carry through a full extension of James' work to produce a geometric model. In his dissertation, written under the supervision of E. Calabi, Jim had carried out a detailed study of the homology of symmetric products. Motivated in part by this dissertation work, Jim set himself the task of constructing full geometric analogues for James' models in the case of iterated loop spaces. He succeeded completely, using extremely clever combinatorial constructions, and as a consequence succeeded in describing $H_*(\Omega^k\Sigma^k X, Z/pZ)$ as a functor of $H_*(X, Z/pZ)$. This piece of work was a fundamental advance in homotopy theory, and it is of a great deal of use to this day.

This work supported in part by a grant from the National Science Foundation.

II. Spherical Fibration Theory

It has been well know since the work of Steenrod that vector bundles over a fixed base space can be *classified* by maps to a classifying space BO(n), which is constructed as a direct limit of Grassmannian manifolds of n-planes in high dimensional Euclidean spaces. This statement means that isomorphism classes of vector bundles over a base space X can be identified with $[X, BO(n)]$, the collection of homotopy classes of maps from X to $BO(n)$, and that the correspondence goes via $f \to f^*\xi_n$, where ξ_n is a certain *universal bundle* over $BO(n)$. The cohomology ring of $BO(n)$ becomes and important object in bundle classification, since homotopy classes of maps can often be distinguished by their behavior on cohomology. For this reason, cohomology classes in $BO(n)$ and other classifying spaces are referred to as *characteristic classes*, since their pullbacks to the cohomology of the base space will be characteristic invariants of the bundle.

For many purposes, though, one is interested in classifying not vector bundles, but rather *spherical fibrations.* A spherical fibration is a fibration, whose fiber has the homotopy type of a sphere. Stasheff [St] showed that by analogy with the case of vector bundles, one can construct classifying spaces BG_n which classify fibrations whose fiber has the homotopy type of the n-sphere. One can also pass to a limit over n to get a space BG which classifies *stable spherical fibrations.* Jim became interested in studying the characteristic classes for spherical fibrations, i.e. the cohomology of the space BG. He succeeded completely in obtaining a description of the mod-2 cohomology ring of all the spaces BG_n [M3]. In order to do this, he first had to study the homology algebra of the space G_n of self equivalences of the n-sphere. This he carried out using his earlier work on the structure of iterated loop spaces. This calculation was then input to an Eilenberg Moore spectral sequence, which he was able to analyze. The end result is that the cohomology ring may be described as

$$H^*(BG, Z/2Z) \cong H^*(BO, Z/2Z) \otimes \Lambda(e_i; i \in I)$$

where the left hand tensor factor denotes the cohomology ring of the union of all the classifying spaces $BO(n)$ and the right hand side denotes an exterior algebra on an infinite set of generators parametrized by a set I.

III. Manifold Classification and Bordism

In the mid 1950's R. Thom [Th] defined the bordism groups of smooth unoriented manifolds. Two smooth n-manifolds M and N are said to be *bordant* if there is a smooth $n+1$ dimensional manifold with boundary W whose boundary is the disjoint union of M and N. This gives an equivalence relation on the collection of all smooth manifolds. Further, the equivalences form an abelian group denoted Ω_n, since one may use the disjoint union operation to add bordism classes of manifolds. Thom asked what the structure of these groups were, and provided an answer via the well-known *Pontrjagin-Thom* construction. He showed that the bordism groups could be identified with homotopy groups of *Thom complexes of vector bundles*, so that the seemingly unapproachable problem of manifold classification up to bordism was reduced to a more concrete question in homotopy theory. Thom then went on to carry out the homotopy theoretic analysis of these Thom complexes, and found an explicit answer for the unoriented bordism groups. The key ingredient was an observation that the cohomology groups of the Thom complexes behaved like free modules over the Steenrod algebra.

In the late 1950's and early 1960's, topologists began the study of bordism of smooth manifolds with additional structure. For instance, one can ask for the classification of oriented manifolds, almost complex manifolds, spin manifolds, etc., up to a bordism W carrying the same structure. These cases were carried out: Wall computed the oriented bordism groups [W1], Milnor the almost complex case [Mi2], and Anderson-Brown-Petersen the spin case [A-B-P]. There are many other possible structures, and much work has gone into their analysis. The method is always to study the homotopy type of a Thom complex of a vector bundle over a classifying space.

By the late 1960's, topologists were becoming interested in studying bordism of manifolds which weren't necessarily smooth. In the earlier work, smoothness was a key ingredient, since the Pontrjagin-Thom construction required the presence of a normal bundle to an embedding as well as a good transversality theory. So, the study of topological manifolds and piecewise linear (PL) manifolds required the development of a theory of normal bundles. This was carried out by Rourke-Sanderson ([R-S1],[R-S2], and [R-S3]) and Milnor [Mi3]. Their work permitted the reduction of topological and PL bordism groups to the homotopy theory of Thom complexes of generalized "bundles" over classifying spaces $BTop$ and BPL.

The homotopy theoretic analysis of these Thom complexes was carried out by Brumfiel-Madsen-Milgram, at the prime 2 [B-M-M]. They completed a 2-step program, the first step being the analysis of the cohomology rings $H^*(BTop, Z/2Z)$ and $H^*(BPL, Z/2Z)$, and the second being the computation of the homotopy groups from this information, using the Thom isomorphism. It turns out that the first step was in this case almost the whole story, that the computation of the homotopy groups from the cohomology was almost immediate. The difficult portion was the computation of the cohomology rings. This was carried out using the two fibrations

$$G/Top \longrightarrow BTop \longrightarrow BG$$

and

$$G/PL \longrightarrow BPL \longrightarrow BG$$

G/Top and G/PL were studied by Sullivan [Su], and BG had been studied by Milgram [M3], as we saw above. Sullivan had found that G/Top and G/PL could be described as products of simple spaces, either Eilenberg-MacLane spaces or two stage Postnikov systems. The end result was the complete description of the topological and piecewise linear unoriented bordism. Madsen and Milgram carried this work further to obtain the oriented versions of these results [MaMi1].

IV. Surgery

Surgery theory is a method developed by Browder [B2] and Novikov [N] in the simply connected case, and by Wall [W2] in the non-simply connected case, for constructing and classifying manifold structures on a topological space. It has been extremely successful in addressing a number of interesting geometric problems. Jim has done a great deal of work in this area, and we will first summarize the method.

The question we are asking about a topological space is if it has the homotopy type of a compact manifold without boundary, and if so to classify the distinct such manifolds, where manifolds with the given homotopy type are distinct if they are not homeomorphic. The first observation is that because of the Poincaré Duality Theorem, the cohomology ring of a space which has the homotopy type of a closed manifold must satisfy the Poincaré duality. Such spaces are referred to as *Poincaré* *duality spaces.* It turns out that although Poincaré duality spaces do not come with a tangent bundle, they do have a *Spivak normal fibration* (see [Sp]). The Spivak normal fibration is a stable class of spherical fibrations, in other words it gives a homotopy class of maps from the space in question to the classifying space BG described above. If the space is known to be a smooth manifold, then the Spivak normal fibration can be identified with the unit sphere bundle in the normal bundle to a smooth embedding of the manifold in Euclidean space. This means that for a Poincaré duality space to be the underlying space of a smooth manifold, there must be a reduction of the Spivak spherical fibration to a vector bundle. This means that in the diagram

$$\begin{array}{ccc} & & BO \\ & & \downarrow \\ X & \xrightarrow{\text{Spivak}} & BG \end{array}$$

there must be a lift of the classifying map to BO. This is a real restriction, which fails for many Poincaré duality spaces, as can be verified using Jim Milgram's computation of $H^*(BG)$. Supposing that we have such a reduction, we can construct one of the objects which surgery theory deals with, a *degree one normal map.* A degree one normal map is a map $f : M \longrightarrow X$ from a smooth manifold M to X, which induces an isomorphism on cohomology in the top non-zero dimension, together with a vector bundle ξ over X, and a stable isomorphism of vector bundles from ν_M, the stable normal bundle of M, to the pullback bundle $f^*\xi$. The question asked by surgery theory is whether a degree one normal map can be modified (using surgeries) to one which is a homotopy equivalence. Surgeries are handle addition operations performed to M, in such a way that the map f and the bundle isomorphisms may be extended over the handles. The result of a surgery is a new degree one normal map, which is *normally bordant* to the original map, in the sense that there is an $(n+1)$-dimensional manifold W with boundary, with a map to X, and with an identification of the stable normal bundle to W with the pullback of ξ, so that the boundary of W is the disjoint union of the two domain manifolds, and so that the restriction of the "bundle data" is compatible with the bundle data on the two pieces. Browder and Novikov arrived at very explicit answers concerning when a degree one normal map may be modified by surgeries to one which is a homotopy equivalence. They showed the following in the simply connected case.

1. For any degree one normal map involving manifolds and Poincaré duality spaces of odd dimension greater than or equal to 5, one can always modify the map by surgeries to obtain a homotopy equivalence.
2. If the dimension n is a multiple of 4, and greater than or equal to 5, a degree one normal map can be modified by surgeries to a homotopy equivalence if and only if the difference $\textbf{signature}(X) - \textbf{signature}(M)$ is equal to zero. (Poincaré duality spaces have signatures just as manifolds do, since they have a non-singular middle dimensional form)
3. If n is of the form $4k+2$, and is greater than or equal to 5, then there is a $Z/2Z$-valued obstruction, referred to as the *Kervaire invariant*, so that the degree one normal map can be modified by surgeries to a homotopy equivalence if and only if this obstruction vanishes.

C. T. C. Wall in [W2] constructed a similar theory for non-simply connected manifolds. He showed that there are 4-periodic obstruction groups $L_n(\pi_1(X))$, depending only on the fundamental group $\pi_1(X)$, so that for a degree one normal map of dimension greater than or equal to 5, that map may be modified to a homotopy equivalence if and only if an obstruction in the appropriate obstruction group vanishes.

Here are some of Jim's contributions to this area.

1. Calculations with Brumfiel and Madsen [Ma-Mi2] which provide understanding of the homological behavior of spaces which play a key role in manifold classification. This includes the work on BG, $BTop$, and BPL, as well as the spaces G/PL and G/TOP.
2. Calculations of L-groups, joint with Hambleton [H-M1], [H-M2] and Carlsson [Ca-M].
3. Work with Hambleton, Taylor, and Williams [H-M-T-W] on the "oozing conjecture", which provides severe restrictions on which surgery obstructions can actually appear on degree one normal maps involving closed manifolds.
4. Applications of surgery and algebraic K-theory to the study of the "topological space form problem", which we will now discuss in more detail.

V. The Topological Space Form Problem

This problem is a particularly beautiful application of non-simply connected surgery theory, to which Jim has made very important contributions. This problem had its origins in the 19th century. The question that was formulated and solved then was, "which finite groups G act freely and linearly on a sphere". A linear action on a sphere is the restriction of a representation to the unit sphere in the representation space under a group invariant inner product. An equivalent problem is to ask, "which finite groups G occur as the fundamental groups of manifolds with constant positive curvature?". Representation theory allows us to give an explicit answer to this question. The following two conditions are necessary and sufficient.

1. All abelian subgroups of G are cyclic. This follows easily since non-cyclic abelian groups admit no free linear actions on spheres, as is easily verified using the character tables.
2. For any two distinct primes p and q, all subgroups of order pq are cyclic. This means that no non-trivial semidirect products of Z/pZ and Z/qZ occur. This is also relatively easy to check, using the character tables of the various non-trivial semidirect products. These conditions are referred to as the *pq-conditions.*

That these conditions are necessary is not hard to check, as we have seen, but the proof that they are also necessary requires a long argument which effectively classifies all groups satisfying these conditions.

C. T. C. Wall [W2] proposed that we remove the linearity requirement on the group action on the sphere, or equivalently that we remove any condition on curvature on the orbit manifold and instead simply require that the universal cover is topologically a sphere. Explicitly, the *topological space form problem* asks, "which finite groups G act freely on the n-sphere?".

The first thing to find out in attacking this question is how many of the necessary conditions which hold in the linear case continue to hold in the topological problem. The first condition, that all abelian subgroups must be cyclic, continues to hold. This is relatively easy to check, since a spectral sequence argument shows that for groups G which act freely on spheres, the cohomology ring $H^*(G, Z)$ must be *periodic*, i.e. that for some d, there are isomorphisms $H^\ell(G,Z) \cong H^{\ell+d}(G,Z)$ for all $\ell > 0$. Since the cohomology rings of all abelian groups are known explicitly, we can readily check that this condition doesn't hold unless the group is cyclic. R. G. Swan [Sw] showed that this condition on the abelian subgroups is sufficient to guarantee the existence of a free action of G on a finite CW-complex X having the homotopy type of a sphere. From the point of view of non-simply connected surgery theory, the orbit space X/G is now a Poincaré Duality space, to which we can apply the techniques of surgery. As for the pq-conditions, they do not hold in general, as was observed by T. Petrie [P] who produced (using surgery theory) free actions of non-trivial semidirect products of order pq on spheres. Milnor [Mi1] proved, though, that all the $2p$-conditions hold, i.e. that subgroups of order $2p$, where p is an odd prime, must be cyclic.

I. Madsen, C. B. Thomas, and C. T. C. Wall ([Ma-T-W1],[Ma-T-W2]) showed that the conditions on abelian subgroups being cyclic, together with the $2p$-conditions for all odd primes p, imply that there is a free action of G on a sphere. This theorem is one of the important achievements of surgery theory. However, it leaves open the question about which spheres G acts on. In many cases they show that G does in fact act on S^d, where d is the smallest period for the cohomology, but they are not able to resolve this question in general. Here, a more delicate analysis is required.

To describe this analysis, we first have to return to the theorem of Swan, which says that for G with only cyclic abelian subgroups, there is a finite CW-complex on which G acts. From the spectral sequence analysis and arguments about periodic cohomology, it is clear that the dimension of the sphere must be of the form $dk-1$, where d is the period in the cohomology, and where k is an arbitrary positive integer. However, the complex which Swan constructs cannot always be made d-dimensional. There is always a complex X (perhaps not finite) which has the homotopy type of S^d, so that the orbit complex has finite cohomology (and hence looks cohomologically like a finite complex), but it can't necessarily be made homotopy equivalent to a finite complex. There is a *Wall finiteness obstruction* (depending on the choice of complex X) in the algebraic K-group $K_0(Z[G])$ which must vanish if X is to have the homotopy type of a finite complex.

Jim ([M4],[Da-M]) showed that there are indeed situations in which G cannot act in the period dimension d, as a result of the fact that X cannot be chosen with vanishing finiteness obstruction. He showed that among the groups $Q(8a,b,c)$, there are values of a, b, and c for which the finiteness obstruction cannot be made to vanish, and so that these groups do not act in the period dimension. $Q(8a,b,c)$ is a group which fits into an extension

$$1 \longrightarrow C \longrightarrow Q(8a,b,c) \longrightarrow \overline{Q} \longrightarrow 1$$

where $\overline{Q}$ is a generalizes quaternion subgroup of $SU(2)$. He also showed that there are values of a, b, and c where the work of Madsen-Thomas-Wall does not guarantee an action in the period, but where an analysis of the finiteness obstruction will show that an action exists. This has shown the subtlety of the question, and Jim has shown how to perform the analysis of these delicate invariants. Jim (and Ib Madsen, independently) have also shown that even in situations where the finiteness obstruction does not preclude the existence of an action in the period dimension, the surgery obstruction may be non-zero, and so that an action in the period dimension will not exist.

VI. Atiyah Jones Conjecture

In the late 1970's, M. F. Atiyah and J. D. S. Jones [A-J] studied the moduli space solutions to the Yang-Mills equations on the space of connections on principal $SU(2)$-bundles on S^4. These solutions are named *instantons*, for reasons arising in physics. For each value of a positive integer d there is a principal $SU(2)$ bundle ξ_d with Chern class d, and we can consider the moduli space M_d of solutions to the self-duality equations on the space of connections on this bundle. We can also consider the space C_d of gauge equivalence classes of *all* connections on ξ_d, and we have the evident inclusion $i_d : M_d \longrightarrow C_d$. Atiyah and Jones found that there is an integer valued function $N(d)$ so that the induced map $H_*(i_d)$ is surjective for $* < N(d)$, and they went on to conjecture that $H_*(i_d)$ is in fact an isomorphism for $*$ less than some function $N(d)$.

Taubes [Ta] and Gravesen [G] proved a result in this direction. They observed that the spaces M_d fit together into a directed system

$$\cdots \longrightarrow M_d \longrightarrow M_{d+1} \longrightarrow M_{d+2} \longrightarrow \cdots$$

and that we therefore obtain a map on the direct limits

$$\lim M_d \longrightarrow \lim C_d.$$

They showed that this map is a homotopy equivalence. Their result is only a result about the limits, though, it does not show that this ever induces an isomorphism on homotopy groups for any particular value of d.

Jim, in collaboration with C. Boyer, J. Hurtubise, and B. Mann, was able to prove the conjecture of Atiyah-Jones, with $N(d) = \lfloor \frac{d}{2} \rfloor - 2$ [B-H-M-M]. The method used the work of Taubes-Gravesen to reduce the result to a stability argument for the maps $M_d \longrightarrow M_{d+1}$. The corresponding stability statement for the spaces C_d is immediate, since we have equivalences $C_d \cong \Omega^3_d S^3$, where the subscript denotes "degree d component", and the inclusions $C_d \longrightarrow C_{d+1}$ are given by loop sum with a degree one map. This conjecture had been the subject of intense work for a number of years, and its solution is one of the big achievements in the subject over the last 20 years.

VII. Cohomology of Simple Groups

Jim has had a large impact on the area of mathematics centered around the cohomology of finite groups. From his early days he acquired a hands-on knowledge of the cohomology of symmetric products (his thesis topic) and later the cohomology of infinite loop spaces. Along the way he developed a unique geometric insight into the cohomology of the symmetric and alternating groups, connecting Nakaoka's celebrated work with loop space techniques and invariant theory. Jim was able to use these methods to devise a method for computing the mod 2 cohomology of the finite symmetric groups. These computations and the detection arguments implemented by Jim were very influential for Quillen's subsequent ground-breaking work on group cohomology and the Adams conjecture.

Many years later (around 1988), Jim returned to his interests in finite group cohomology and he became an important catalyst for extensive ongoing interactions between algebraists and topologists. In a series of papers (many of them joint with A. Adem), Jim outlined an approach for computing the mod 2 cohomology of many of the low-rank sporadic simple groups ([A-K-M-U], [A-M-M1], [A-M-M2], [A-M1], [A-M3], [A-M4], [A-M5], [A-M6], [Ca-M-M], [M5], [M6], [M7]) . Along the way these examples have provided testing grounds for a number of theorems and conjectures. These calculations have radically altered the nature of research in this area.

As a result of this research, Adem and Milgram published the Springer-Verlag Grundlehren text Cohomology of Finite Groups, [A-M2] which will be a standard reference in the subject for many years to come.

VIII. Mathematics Education

Over the last few years, Jim has in addition to his research work become interested in elementary and secondary mathematics education. He has been instrumental in the introduction of Content Standards in California, as well as in the development of a Framework for their implementation. In addition, he continues to play a important role as an advisor to the state of California on mathematics education, and he plays a leading role in the textbook adoption process in California. This work has had profound consequences for millions of children in California as well as throughout the country.

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A renormalized Riemann-Roch formula and the Thom isomorphism for the free loop space

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ABSTRACT. Let E be a circle-equivariant complex-orientable cohomology theory. We show that the fixed-point formula applied to the free loop space of a manifold X can be understood as a Riemann-Roch formula for the quotient of the formal group of E by a free cyclic subgroup. The quotient is not representable, but (locally at p) its p-torsion subgroup is, by a p-divisible group of height one greater than the formal group of E.

I believe in the fundamental interconnectedness of all things.
—Dirk Gently [**Ada88**]

1. Introduction

Let $\mathbb{T}$ denote the circle group, and, if X is a compact smooth manifold, let $\mathcal{L}X \stackrel{\text{def}}{=} C^\infty(\mathbb{T}, X)$ denote its free loop space. The group $\mathbb{T}$ acts on $\mathcal{L}X$, and the fixed point manifold is again X, considered as the subspace of constant loops. In the 1980's, Witten showed that the fixed-point formula in ordinary equivariant cohomology, applied to the free loop space $\mathcal{L}X$ of a spin manifold X, yields the index of the Dirac operator (i.e. the $\hat{A}$-genus) of X—a fundamentally K-theoretic quantity [**Ati85**]. He also applied the fixed-point theorem in equivariant K-theory to a Dirac-like operator on $\mathcal{L}X$ to obtain the elliptic genus and "Witten genus" of X [**Wit88**]—quantities associated with elliptic cohomology.

Among homotopy theorists, these developments generated considerable excitement. The chromatic program organizes the structure of finite stable homotopy types, locally at a prime p, into layers indexed by nonnegative integers. The nth layer is detected by a family of cohomology theories $\mathcal{E}_n$; rational cohomology, K-theory, and elliptic cohomology are detecting theories for the first three layers [**Mor85, DHS88, HS98**].

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The geometry and analysis related to rational cohomology and K-theory are reasonably well-understood, but for $n \geq 2$ and for elliptic cohomology in particular, very little is known. Witten's work provides a major suggestion: for $n = 1$ and $n = 2$ his analysis gives a correspondence

$$\begin{matrix}\text{analysis underlying } \mathcal{E}_n \\ \text{applied to } X\end{matrix} \leftrightarrow \begin{matrix}\text{analysis underlying } \mathcal{E}_{n-1} \\ \text{applied to } \mathcal{L}X.\end{matrix} \tag{1.1}$$

This paper represents our attempt to understand why Witten's procedure appears to connect the chromatic layers in the manner of (1.1). To do this we consider very generally the fixed-point formula attached to a complex-oriented theory E with formal group law F. We recall that for $n > 0$, such a theory detects chromatic layer n if the formal group law F has height n.

Our first result is that the fixed-point formula of a suitable equivariant extension of E (Borel cohomology is fine, as is the usual equivariant K-theory) applied to the free loop space yields a formula which is *identical* to the Riemann-Roch formula for the quotient $F/(\hat{q})$ of the formal group law F by a free cyclic subgroup $(\hat{q})$ (compare formulae (3.4) and (4.5)).

The quotient $F/(\hat{q})$ is not a formal group, so to understand its structure, we work p-locally and study its p-torsion subgroup $F/(\hat{q})[p^\infty]$. We construct a group $\mathrm{Tate}(F)$ with a canonical map

$$\mathrm{Tate}(F) \to F/(\hat{q}),$$

which induces an isomorphism of torsion subgroups in a suitable setting. Our second result is that the group $\mathrm{Tate}(F)[p^\infty]$ is a p-divisible group, fitting into an extension

$$F[p^\infty] \to \mathrm{Tate}(F) \to \mathbb{Q}_p/\mathbb{Z}_p$$

of p-divisible groups. If the height of F is n, then the height of $\mathrm{Tate}(F)[p^\infty]$ is $n+1$, but its étale quotient has height 1. In a sense we make precise in §5.3, it is the universal such extension.

Thus the fixed-point formula on the free loop space interpolates between the chromatic layers in the same way that p-divisible groups of height $n+1$ with étale quotient of height 1 interpolate between formal groups of height n and formal groups of height $n+1$. This is discussed in more detail, from the homotopy-theoretic point of view, in our earlier paper [**AMS98**] with Hal Sadofsky; this paper is a kind of continuation, concerned with analytic aspects of these phenomena. We show that Witten's construction in rational cohomology produces K-theoretic genera because of the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1 \tag{1.2}$$

expressing the multiplicative group (K-theory) as the quotient of the additive group (ordinary cohomology) by a free cyclic subgroup; while his work in K-theory produces elliptic genera because of the exact sequence

$$0 \to q^{\mathbb{Z}} \to \mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} \to 1 \tag{1.3}$$

(where q is a complex number with $|q| < 1$), expressing the Tate elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ as the quotient of the multiplicative group by a free cyclic subgroup.

These analytic quotients have already been put to good use in equivariant topology. Grojnowski constructs from equivariant ordinary cohomology a complex $\mathbb{T}$-equivariant elliptic cohomology using the elliptic curve $\mathbb{C}/\Lambda$ which is the quotient of the complex plane by a lattice; and Rosu uses Grojnowski's functor to give a striking conceptual proof of the rigidity of the elliptic genus. Grojnowski's ideas applied to the multiplicative sequence (1.3) give a construction of complex $\mathbb{T}$-equivariant elliptic cohomology based on equivariant K-theory; details will appear elsewhere. Completing this circle, Rosu has used the quotient (1.2) to give a construction of complex equivariant K-theory [**Gro94, Ros99, RK99**].

Several of the formulae in this paper involve formal infinite products; see for example (3.4) and (4.5). On the fixed-point formula side, the source of these is the Euler class of the normal bundle ν of X in $\mathcal{L}X$ (3.2). From this point of view, the problem is that the bundle ν is infinite-dimensional, so it does not have a Thom spectrum in the usual sense. However, ν has a highly nontrivial circle action, which defines a locally finite-dimensional filtration by eigenspaces. Following the program sketched in [**CJS95**], we construct from this filtration a Thom *pro-spectrum*, whose Thom class is the infinite product.

In the particular cases of the additive and multiplicative formal groups ($n = 1, 2$ above), one can also control the infinite products by replacing them with products which converge to holomorphic functions on $\mathbb{C}$; this construction of elliptic functions goes back to Eisenstein. We are grateful to Kapranov for pointing out to us that Eistenstein considered the the analogous problem for $n > 2$. In [**Eis44**] he described the difficulty of interpreting such infinite products. He went on to hint that he perceived a useful approach, and concluded the following.

> Die Functionen, zu welchen man auf diesen Wege geführt wird, scheinen sehr merkwürdige Eigenschaften zu besitzen; sie eröffnen ein Feld, auf dem sich Stoff zu den reichhaltigsten Untersuchungen darbietet, und welches der eigentliche Grund und Boden zu sein scheint, auf welchem die schwierigsten Theile der Analysis und Zahlentheorie ineinander greifen.

1.1. Formal group schemes. In this paper (especially in section 5) we shall consider formal schemes in the sense of [**Str99, Dem72**]. A *formal scheme* is a filtered colimit of affine schemes. For example the "formal line"

$$\hat{\mathbb{A}}^1 \overset{\text{def}}{=} \operatorname*{colim}_n \operatorname{spec} \mathbb{Z}[x]/x^n$$

is a formal scheme. Note that an affine scheme is a formal scheme in a trivial way. An important feature of this category which we shall use is that it has finite products. For example,

$$\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1 = \operatorname{colim} \operatorname{spec}\big(\mathbb{Z}[x]/(x^n) \otimes \mathbb{Z}[y]/(y^m)\big).$$

In particular a *formal group scheme* means an abelian group in the category of formal schemes. A formal group scheme whose underlying formal scheme is isomorphic to the formal scheme $\hat{\mathbb{A}}^1$ is called a commutative one-dimensional formal Lie group. We shall simply call it a *formal group.*