Graduate Texts in Mathematics

John L. Kelley

General Topology

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PREFACE

This book is a systematic exposition of the part of general topology which has proven useful in several branches of mathematics. It is especially intended as background for modern analysis, and I have, with difficulty, been prevented by my friends from labeling it: What Every Young Analyst Should Know.

The book, which is based on various lectures given at the University of Chicago in 1946–47, the University of California in 1948–49, and at Tulane University in 1950–51, is intended to be both a reference and a text. These objectives are somewhat inconsistent. In particular, as a reference work it offers a reasonably complete coverage of the area, and this has resulted in a more extended treatment than would normally be given in a course. There are many details which are arranged primarily for reference work; for example, I have taken some pains to include all of the most commonly used terminology, and these terms are listed in the index. On the other hand, because it is a text the exposition in the earlier chapters proceeds at a rather pedestrian pace. For the same reason there is a preliminary chapter, not a part of the systematic exposition, which covers those topics requisite to the main body of work that I have found to be new to many students. The more serious results of this chapter are theorems on set theory, of which a systematic exposition is given in the appendix. This appendix is entirely independent of the remainder of the book, but with this exception each part of the book presupposes all earlier developments.

There are a few novelties in the presentation. Occasionally the title of a section is preceded by an asterisk; this indicates that the section constitutes a digression. Other topics, many of equal or greater interest, have been treated in the problems. These problems are supposed to be an integral part of the discussion. A few of them are exercises which are intended simply to aid in understanding the concepts employed. Others are counter examples, marking out the boundaries of possible theorems. Some are small theories which are of interest in themselves, and still others are introductions to applications of general topology in various fields. These last always include references so that the interested reader (that elusive creature) may continue his reading. The bibliography includes most of the recent contributions which are pertinent, a few outstanding earlier contributions, and a few "cross-field" references.

I employ two special conventions. In some cases where mathematical content requires "if and only if" and euphony demands something less I use Halmos' "iff." The end of each proof is signalized by ▌. This notation is also due to Halmos.

J. L. K.

Berkeley, California
February 1, 1955

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It is a pleasure to acknowledge my indebtedness to several colleagues.

The theorems surrounding the concept of even continuity in chapter 7 are the joint work of A. P. Morse and myself and are published here with his permission. Many of the pleasanter features of the appended development of set theory are taken from the unpublished system of Morse, and I am grateful for his permission to use these; he is not responsible for inaccuracies in my writing. I am also indebted to Alfred Tarski for several conversations on set theory and logic.

I owe thanks to several colleagues who have read part or all of the manuscript and made valuable criticisms. I am particularly obliged to Isaku Namioka, who has corrected a grievous number of errors and obscurities in the text and has suggested many improvements. Hugo Ribeiro and Paul R. Halmos have also helped a great deal with their advice.

Finally, I tender my very warm thanks to Tulane University and to the Office of Naval Research for support during the preparation of this manuscript. This book was written at Tulane University during the years 1950–52; it was revised in 1953, during tenure of a National Science Foundation Fellowship and a sabbatical leave from the University of California.

J. L. K.

April 21, 1961

A number of corrections have been made in this printing of the text. I am indebted to many colleagues, and especially to Krehe Ritter, for bringing errors to my attention.

J. L. K.

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Chapter 0

PRELIMINARIES

The only prerequisites for understanding this book are a knowledge of a few of the properties of the real numbers and a reasonable endowment of that invaluable quality, mathematical maturity. All of the definitions and basic theorems which are assumed later are collected in this first chapter. The treatment is reasonably self-contained, but, especially in the discussion of the number system, a good many details are omitted. The most profound results of the chapter are theorems of set theory, of which a systematic treatment is given in the appendix. Because the chapter is intended primarily for reference it is suggested that the reader review the first two sections and then turn to chapter one, using the remainder of the chapter if need arises. Many of the definitions are repeated when they first occur in the course of the work.

SETS

We shall be concerned with sets and with members of sets. "Set," "class," "family," "collection," and "aggregate" are synonymous,* and the symbol ε denotes membership. Thus $x \varepsilon A$ if and only if x is a member (an element, a point) of A. Two sets are identical iff they have the same members, and equality is

* This statement is not strictly accurate. There are technical reasons, expounded in the appendix, for distinguishing between two different sorts of aggregates. The term "set" will be reserved for classes which are themselves members of classes. This distinction is of no great importance here; with a single non-trivial exception, each class which occurs in the discussion (prior to the appendix) is also a set.

always used to mean identity. Consequently, $A = B$ if and only if, for each x, $x \in A$ when and only when $x \in B$.

Sets will be formed by means of braces, so that $\{x: \cdots$ (*proposition about* x) $\cdots\}$ is the set of all points x such that the proposition about x is correct. Schematically, $y \in \{x: \cdots$ (*proposition about* x) $\cdots\}$ if and only if the corresponding proposition about y is correct. For example, if A is a set, then $y \in \{x: x \in A\}$ iff $y \in A$. Because sets having the same members are identical, $A = \{x: x \in A\}$, a pleasant if not astonishing fact. It is to be understood that in this scheme for constructing sets "x" is a dummy variable, in the sense that we may replace it by any other variable that does not occur in the proposition. Thus $\{x: x \in A\} = \{y: y \in A\}$, but $\{x: x \in A\} \neq \{A: A \in A\}$.

There is a very useful rule about the construction of sets in this fashion. If sets are constructed from two different propositions by the use of the convention above, and if the two propositions are logically equivalent, then the constructed sets are identical. The rule may be justified by showing that the constructed sets have the same members. For example, if A and B are sets, then $\{x: x \in A \textit{ or } x \in B\} = \{x: x \in B \textit{ or } x \in A\}$, because y belongs to the first iff $y \in A$ or $y \in B$, and this is the case iff $y \in B$ or $y \in A$, which is correct iff y is a member of the second set. All of the theorems of the next section are proved in precisely this way.

SUBSETS AND COMPLEMENTS; UNION AND INTERSECTION

If A and B are sets (or families, or collections), then A is a **subset (subfamily, subcollection)** of B if and only if each member of A is a member of B. In this case we also say that A is **contained** in B and that B **contains** A, and we write the following: $A \subset B$ and $B \supset A$. Thus $A \subset B$ iff for each x it is true that $x \in B$ whenever $x \in A$. The set A is a **proper subset** of B (A is properly contained in B and B properly contains A) iff $A \subset B$ and $A \neq B$. If A is a subset of B and B is a subset of C, then clearly A is a subset of C. If $A \subset B$ and $B \subset A$, then $A = B$, for in this case each member of A is a member of B and conversely.

The **union (sum, logical sum, join)** of the sets A and B, written $A \cup B$, is the set of all points which belong either to A or to B; that is, $A \cup B = \{x: x \in A \text{ or } x \in B\}$. It is understood that "or" is used here (and always) in the non-exclusive sense, and that points which belong to both A and B also belong to $A \cup B$. The **intersection (product, meet)** of sets A and B, written $A \cap B$, is the set of all points which belong to both A and B; that is, $A \cap B = \{x: x \in A \text{ and } x \in B\}$. The **void set (empty set)** is denoted 0 and is defined to be $\{x: x \neq x\}$. (Any proposition which is always false could be used here instead of $x \neq x$.) The void set is a subset of every set A because each member of 0 (there are none) belongs to A. The inclusions, $0 \subset A \cap B \subset A \subset A \cup B$, are valid for every pair of sets A and B. Two sets A and B are **disjoint,** or **non-intersecting,** iff $A \cap B = 0$; that is, no member of A is also a member of B. The sets A and B **intersect** iff there is a point which belongs to both, so that $A \cap B \neq 0$. If $\mathcal{A}$ is a family of sets (the members of $\mathcal{A}$ are sets), then $\mathcal{A}$ is a **disjoint family** iff no two members of $\mathcal{A}$ intersect.

The **absolute complement** of a set A, written $\sim A$, is $\{x: x \notin A\}$. The **relative complement** of A with respect to a set X is $X \cap \sim A$, or simply $X \sim A$. This set is also called the **difference** of X and A. For each set A it is true that $\sim\sim A = A$; the corresponding statement for relative complements is slightly more complicated and is given as part of 0.2.

One must distinguish very carefully between "member" and "subset." The set whose only member is x is called **singleton** x and is denoted $\{x\}$. Observe that $\{0\}$ is not void, since $0 \in \{0\}$, and hence $0 \neq \{0\}$. In general, $x \in A$ if and only if $\{x\} \subset A$.

The two following theorems, of which we prove only a part, state some of the most commonly used relationships between the various definitions given above. These are basic facts and will frequently be used without explicit reference.

1 Theorem *Let A and B be subsets of a set X. Then $A \subset B$ if and only if any one of the following conditions holds:*

$$A \cap B = A, \quad B = A \cup B, \quad X \sim B \subset X \sim A,$$

$$A \cap X \sim B = 0, \quad \text{or} \quad (X \sim A) \cup B = X.$$