Texts and Monographs in physics

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The Quantum Mechanical Few - Body Problem

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With 17 Figures

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

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ISBN 3-540-12587-6 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12587-6 Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging in Publication Data. Glöckle, Walter, 1939–. The quantum mechanical few-body problem. (Texts and monographs in physics). Bibliography: p. Includes index. 1. Few-body problem. 2. Quantum theory. I. Title. II. Series. QC174.17.P7G58 1983 530.1'2 83-10325

Typesetting: K + V Fotosatz, Beerfelden.
Offset printing and bookbinding: Konrad Triltsch, Graphischer Betrieb, Würzburg
2153/3130-543210

Preface

Few-body systems are both technically relatively simple and physically non-trivial enough to test theories quantitatively. For instance the He-atom played historically an important role in verifying predictions of QED. A similar role is contributed nowadays to the three-nucleon system as a testing ground for nuclear dynamics and maybe in the near future to few-quark systems. They are also often the basic building blocks for many-body systems like to some extent nuclei, where the *real many-body* aspect is not the dominant feature.

The presentation of the subject given here is based on lectures held at various places in the last ten years. The selection of the topics is certainly subjective and influenced by my own research interests. The content of the book is simply organized according to the increasing number of particles treated. Because of its conceptual simplicity single particle motion is very suitable for introducing the basic elements of scattering theory. Using these elements the two-body system is treated for the specific case of two nucleons, which is of great importance in the study of the nuclear interaction. Great space is devoted to the less trivial few-body system consisting of three particles. Again physical examples are taken solely from nuclear physics. Finally the four-particle system is discussed so as to familiarize the reader with the techniques required for the formulations of n-bodies in general. One of the aims of the n-body connected kernel formulations is to put conventional, intuitively invented nuclear models and reaction theories on a firm basis. Though there are already promising insights available, the break-through has apparently not yet been found and the natural and desired extension of the matter developed here is still on the "second sheet".

In order not to overload the content of these introductory notes and partially because of existing presentations certain techniques and subjects are not dealt with. These are variational methods, the use of hyperspherical harmonics, the elaboration of finite rank approximations of t-operators and kernels (which played and still play an important role), the very interesting problem of formulating a relativistic theory for n particles, and the whole dynamical problem of nuclear forces which includes the very successul recent solution of few-body Bethe-Salpeter equations.

In the techniques and subject treated there exists a large amount of publications. We would like to apologize to those authors whose work is not directly or sufficiently well mentioned. There are several reviews and articles related to our subject. Besides special monographs on few-body systems we refer the reader also to some books which are closely related. An important source of information are the proceedings of the international few-body conferences held up to now. All these sources are cited at the end.

The book is written for students and does not require more than a basic course in QM. It emphasizes also the practical points of view and will hopefully be profitable to some researchers working in that field as well.

This work would not have been undertaken without the continuous stimulation by Professor Hélio T. Coelho. I am very thankful to him and for his kind hospitality which he extended to me at his institute in Recife, where parts of the notes have been written. Dr. R. Brandenburg eradicated my major blunders in English and helped me in some parts to clarify the presentation, for which I thank him very much. Last but not least I want to thank Mrs. Kächele and Mrs. Walter, for their skill and patience in transcribing successfully my handwriting into a legible form.

Bochum, January 1983 *W. Glöckle*

Contents

1. Elements of Potential Scattering Theory

Scattering of a particle by a potential is a simple physical picture but rich enough to introduce such basic concepts of scattering theory as Möller wave operators, in- and outgoing particle flux, unitarity, *S*-, *T*- and *K*-matrices, Lippmann-Schwinger equations, *S*-matrix pole trajectories, criteria for convergence or divergence of Neumann series, etc. Therefore the first chapter is basic and following ones use the language developed here, while enriching and extending it according to the increase of possible physical processes for two and more particles.

1.1 The Möller Wave Operator

Let us regard the scattering of a particle by a potential. We assume that the potential drops towards zero outside a certain bounded domain *D* in space. Initially while approaching *D*, the particle moves freely with a certain momentum. As it crosses *D* it will experience a force which classically would bend the initial straight line trajectory. Having left *D* the particle again moves freely but with a final momentum which can be different from the initial one. It will be the task of Chap. 1 to develop techniques for answering the question of how to find the probability for the change in momentum induced by the potential.

To describe the initial state of free motion outside *D* we have to localize the particle. Let us choose a wave packet $\psi_0(x, t)$, which obeys the time dependent Schrödinger equation

$$H_0 \psi_0(x, t) = \mathrm{i}\,\frac{\partial \psi_0(x, t)}{\partial t} \tag{1.1}$$

with the free Hamilton operator

$$H_0 = -\frac{1}{2m}\nabla^2 . \tag{1.2}$$

We put $\hbar = c = 1$. Then units for energy and length convenient for nuclear physics result from $\hbar c = 197.33$ MeV fm.

Clearly $\psi_0(x, t)$ will be of the general form

$$\psi_0(x, t) = \frac{1}{(2\pi)^{3/2}} \int dq \exp[\mathrm{i}(qx - E_q t)] f_0(q), \tag{1.3}$$

which is a superposition of momentum eigenstates

$$\psi_q^0(x) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}qx} \tag{1.4}$$

with the energies $E_q = q^2/2m$. In a scattering process the momentum distribution $f_0(q)$ will be peaked at an initial momentum q_i.

For example regard

$$f_0(q) = \frac{1}{b^{3/2}} \left(\frac{2}{\pi}\right)^{3/4} \exp[-(q - q_i)^2/b^2]. \tag{1.5a}$$

The quantity b measures the momentum distribution in the beam. It is a simple exercise to evaluate in that case the integral (1.3). The result is

$$\psi_0(x, t) = \frac{1}{(2\pi)^{3/2}} \exp[\mathrm{i}(q_\mathrm{i}x - E_{q_\mathrm{i}}t)] \times (2\pi)^{3/4} b^{3/2} \frac{\exp\left[-\dfrac{b^2}{4} \dfrac{\left(x - t\dfrac{q_\mathrm{i}}{m}\right)^2}{1 + \mathrm{i}tb^2/2m}\right]}{\left(1 + \dfrac{\mathrm{i}t}{2m} b^2\right)^{3/2}}. \tag{1.5b}$$

Thus we find a plane wave with the central momentum q_i in a region of space of extension $d \sim b^{-1}$. The center of the wave packet travels along the classical path. The spreading of the wave packet is controlled by the parameter

$$\xi = \frac{t}{2m} b^2 = \frac{L}{2q} b^2 \approx \frac{L}{d} \frac{b}{q}. \tag{1.5c}$$

Here we introduced a typical length L between source and detector and the average momentum q of the particle. Under ordinary conditions $\xi \ll 1$ and the spreading is negligible.

As the wave packet approaches D it will feel the potential V and its evolution in time will be governed by the time-dependent Schrödinger equation

$$H\Psi(x,t) = \mathrm{i}\frac{\partial \Psi(x,t)}{\partial t} \tag{1.6}$$

with the full Hamilton operator

$$H = H_0 + V. \tag{1.7}$$

So we face the question, how is $\Psi(x,t)$ linked to $\psi_0(x,t)$ or in other words how can we select out of the many solutions of (1.6) that specific one which develops out of the initial state $\psi_0(x,t)$? A first guess could be to fix Ψ through

$$\Psi(x,t) \to \psi_0(x,t) \quad \text{for} \quad t \to -\infty.$$

This requirement however is too weak, since both wave functions tend pointwise towards zero in that limit, and one cannot distinguish between different initial states ψ_0. In the example (1.5b) ψ_0 tends towards zero pointwise like $|t|^{-3/2}$. This is true in general.

Exercise: Prove that

$$F(t) = \int_0^\infty dq\, q^2 \mathrm{e}^{-\mathrm{i}q^2 t} f(q)$$

tends towards

$$\mathrm{const}/|t|^{3/2} \quad \text{for} \quad |t| \to \infty \quad \text{if} \quad f(0) \neq 0.$$

Hint: use the method of steepest descend [1.1].

Although the wave functions spread out with time, leading to smaller and smaller amplitudes at each point x, their norms

$$\|\Psi(t)\| = \sqrt{\int dx\, |\Psi(x,t)|^2} \tag{1.8}$$

are time independent. Therefore, in order to enforce the equality of Ψ and ψ_0 before the particle reaches D we might require

$$\lim_{t \to -\infty} \|\Psi(t) - \psi_0(t)\| \to 0. \tag{1.9}$$

Then the question becomes, is (1.9) compatible with the time dependent Schrödinger equations (1.1, 6)?

Equations (1.1, 6) tell us

$$\begin{aligned} |\psi_0(t)\rangle &= \exp[-\mathrm{i}H_0(t-t_0)]\,|\psi_0(t_0)\rangle \\ |\Psi(t)\rangle &= \exp[-\mathrm{i}H(t-t_0)]\,|\Psi(t_0)\rangle \end{aligned} \tag{1.10}$$

and we can write (1.9) as

$$\begin{aligned}\| \Psi(t)-\psi_0(t) \| &= \|\exp[-\mathrm{i}H(t-t_0)]\,\Psi(t_0)-\exp[-\mathrm{i}H_0(t-t_0)]\,\psi_0(t_0)\| \\ &= \|\Psi(t_0)-\exp[\mathrm{i}H(t-t_0)]\exp[-\mathrm{i}H_0(t-t_0)]\,\psi_0(t_0)\|. \end{aligned} \tag{1.9a}$$

The second equality follows from the unitarity of $\exp[-\mathrm{i}H(t-t_0)]$. Thus the requirement (1.9), together with the time evolution expressed through the Schrödinger equation, will be

$$|\Psi(t_0)\rangle = \lim_{\tau\to-\infty} \mathrm{e}^{\mathrm{i}H\tau}\mathrm{e}^{-\mathrm{i}H_0\tau}|\psi_0(t_0)\rangle. \tag{1.11}$$

If that limit exists, then (1.11) is a link between $|\Psi\rangle$ and $|\psi_0\rangle$, compatible with the Schrödinger equation. Moreover it gives us a prescription for constructing a specific scattering state at the arbitrary time $t=t_0$ which belongs to a certain choice of initial conditions in the infinite past.

The limit in (1.11) defines the Möller wave operator [1.2]

$$\Omega^{(+)} = \lim_{\tau\to-\infty} (\mathrm{e}^{\mathrm{i}H\tau}\mathrm{e}^{-\mathrm{i}H_0\tau}) \tag{1.12}$$

and (1.11) reads for an arbitrary time t

$$|\Psi(t)\rangle = \Omega^{(+)}|\psi_0(t)\rangle. \tag{1.13}$$

This relation (1.13) is the formal solution of the scattering problem to a specific choice of initial conditions.

Let us now sketch a proof [1.3] for the existence of $\Omega^{(+)}$. The ensemble of wave packets $\psi_0(x,t)$ (t fixed) defines the space accessible to the particle. For square integrable momentum distributions they span a Hilbert space. Thus we have to show that $\Omega^{(+)}$ exists on the whole Hilbert space. Define

$$W(t) = \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t} \tag{1.14}$$

and regard

$$\|(W(t_2)-W(t_1))\,\psi_0(0)\| = \left\|\int_{t_1}^{t_2} dt\,\frac{d}{dt}W(t)\,\psi_0(0)\right\|. \tag{1.14a}$$

The limit (1.12) exists if (1.14a) can be shown to be arbitrarily small if $t_1<t_2<0$ and $|t_2|$ is sufficiently large. Now together with the property of W, namely

$$\frac{d}{dt}W(t) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}Ht}(H-H_0)\,\mathrm{e}^{-\mathrm{i}H_0t} = \mathrm{i}\,\mathrm{e}^{\mathrm{i}Ht}V\mathrm{e}^{-\mathrm{i}H_0t}, \tag{1.15}$$

we can estimate the rhs as

$$\left\|\int_{t_1}^{t_2} dt \frac{d}{dt} W(t)\,\psi_0(0)\right\| \leqslant \int_{t_1}^{t_2} dt \left\|\frac{d}{dt} W(t)\,\psi_0(0)\right\|$$

$$= \int_{t_1}^{t_2} dt \|e^{iHt} V\psi_0(t)\| = \int_{t_1}^{t_2} dt \|V\psi_0(t)\| . \tag{1.16}$$

Then using the bound

$$|\psi_0(x,t)| \leqslant \frac{c_1}{c_2 + |t|^{3/2}} \tag{1.17}$$

we end up with

$$\|[W(t_2) - W(t_1)]\,\psi_0(0)\| \leqslant \|V\| \int_{t_1}^{t_2} dt \frac{c_1}{c_2 + |t|^{3/2}} \leqslant c\|V\| . \tag{1.18}$$

Thus provided the potential has a finite norm

$$\|V\|^2 = \int dx\; V^2(x) < \infty \tag{1.19}$$

the Möller wave operator $\Omega^{(+)}$ defined in (1.13) exists. In fact even weaker conditions on V guarantee [1.3] the existence of $\Omega^{(+)}$. The potential has only to be locally square integrable and to decrease faster than the Coulomb potential at infinity.

The result achieved up to now is hardly surprising. We have only formulated and verified everyones expectation that the scattering solutions of the time dependent Schrödinger equation can be specified by certain initial conditions in the infinite past provided the potential is not too long range [see (1.19)]. In addition we have found a certain operator, $\Omega^{(+)}$, which maps the unperturbed initial state $|\psi_0\rangle$ into the complete state $|\Psi\rangle$.

The result (1.12) and (1.13) is not yet a practical one. The standard method of proceeding [1.4] is to reformulate it by using the relation:

$$\lim_{t\to-\infty} f(t) = \lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} dt\, e^{\varepsilon t} f(t) . \tag{1.20}$$

Exercise: Verify (1.20)

We then rewrite (1.13) together with (1.12) as

$$\begin{aligned}|\Psi(0)\rangle &= \lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} dt\, e^{\varepsilon t} e^{iHt} e^{-iH_0 t} |\psi_0(0)\rangle \\ &= \lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} dt\, e^{\varepsilon t} e^{iHt} \int dq\, e^{-iE_q t} |\psi_q^0\rangle f_0(q) \\ &= \lim_{\varepsilon\to 0} \int dq \frac{i\varepsilon}{E_q + i\varepsilon - H} |\psi_q^0\rangle f_0(q) .\end{aligned} \tag{1.21}$$

In this manner we are led to an operator central to scattering theory: the resolvent operator to the Hamiltonian H

$$G(z) \equiv \frac{1}{z-H}. \tag{1.22}$$

Here z should obviously not be in the spectrum of H. Indeed in (1.21) $z = E_q + i\varepsilon$. We shall study properties of G in Sect. 1.3.

It is now tempting to apply G on a momentum eigenstate $|\psi_q^0\rangle$, which is of course not in Hilbert space. We define

$$|\Psi_q^{(+)}\rangle = \lim_{\varepsilon\to 0} \frac{i\varepsilon}{E_q + i\varepsilon - H} |\psi_q^0\rangle \tag{1.23}$$

and verify easily that $|\psi_q^{(+)}\rangle$ is a solution of the stationary Schrödinger equation

$$(H - E_q)|\Psi_q^{(+)}\rangle = 0\,. \tag{1.24}$$

Since these states are not in the Hilbert space special care is needed in their use. Thus (1.23) is the operation by which stationary states to H_0, the momentum eigenstates, are mapped into specific eigenstates of H. The way $|\psi_q^{(+)}\rangle$ incorporates the features of the scattering process will be discussed in Sect. 1.4.

For a specific initial momentum the state $|\Psi_q^{(+)}\rangle$ contains all the information about the scattering process and we get the time dependent state for a general initial momentum distribution by superposition:

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle = \int dq\, |\Psi_q^{(+)}\rangle e^{-iE_q t} f_0(q)\,. \tag{1.25}$$

1.2 The Cross Section

The main result of the last section, (1.25) together with (1.23) allows us to calculate the scattering state at all times. Specifically we can determine the transition amplitude at time t into a state

$$|\psi_{q_f}^0(t)\rangle = e^{-iH_0 t}|\psi_{q_f}^0(0)\rangle$$

of sharp momentum $\boldsymbol{q}_f$:

$$A_{q_f}(t) \equiv \langle \psi_{q_f}^0(t)|\Psi(t)\rangle = \langle \psi_{q_f}^0(0)|e^{iH_0 t}e^{-iHt}|\Psi(0)\rangle. \tag{1.26}$$

Since the cross section is proportional to the transition rate, $(d/dt)\,|A|^2$, we shall also need

$$\dot{A}_{q_f}(t) = -\mathrm{i}\langle \psi^0_{q_f}(0)|e^{\mathrm{i}H_0 t} V e^{-\mathrm{i}Ht}|\Psi(0)\rangle . \tag{1.27}$$

Using (1.25) we find

$$A_{q_f}(t) = \int dq \, \exp[\mathrm{i}(E_{q_f} - E_q)t] \langle \psi^0_{q_f}|\Psi^{(+)}_q\rangle f_0(q) \tag{1.28}$$

and

$$\dot{A}_{q_f}(t) = -\mathrm{i}\int dq \, \exp[\mathrm{i}(E_{q_f} - E_q)t] \langle \psi^0_{q_f}|V|\Psi^{(+)}_q\rangle f_0(q) . \tag{1.29}$$

What are the momentum components $\langle \psi^0_{q_f}|\Psi^{(+)}_q\rangle$ of the stationary scattering state $|\Psi^{(+)}_q\rangle$, which is defined in (1.23)? If we switch off the potential the resolvent operator $G(z)$ turns into

$$G_0(z) \equiv \frac{1}{z - H_0} \tag{1.30}$$

and $|\Psi^{(+)}_q\rangle$ reduces to the momentum eigenstate $|\psi^0_q\rangle$:

$$\lim_{\varepsilon\to 0} \frac{\mathrm{i}\varepsilon}{E_q + \mathrm{i}\varepsilon - H_0}|\psi^0_q\rangle = |\psi^0_q\rangle . \tag{1.31}$$

Then clearly we get

$$\langle \psi^0_{q_f}|\psi^0_q\rangle = \delta^3(q_f - q) , \tag{1.32}$$

which inserted into (1.28) yields just the contribution to $A_{q_f}(t)$ from the unperturbed initial wave packet. How can we explicitly show that part in $|\Psi^{(+)}_q\rangle$? There is an obvious algebraic identity between the two resolvent operators $G_0(z)$ and $G(z)$:

$$G(z) = G_0(z) + G_0(z) V G(z) . \tag{1.33}$$

We use it in (1.23) to separate $|\Psi^{(+)}_q\rangle$ into a free and scattered part:

$$|\Psi^{(+)}_q\rangle = |\psi^0_q\rangle + \lim_{\varepsilon\to 0} \frac{1}{E_q + \mathrm{i}\varepsilon - H_0} V|\Psi^{(+)}_q\rangle . \tag{1.34}$$

Therefore we can express the momentum components of $|\Psi^{(+)}_q\rangle$ as

$$\langle \psi^0_{q_f}|\Psi^{(+)}_q\rangle = \delta^3(q_f - q) + \lim_{\varepsilon\to 0} \frac{\langle \psi^0_{q_f}|V|\Psi^{(+)}_q\rangle}{E_q + \mathrm{i}\varepsilon - E_{q_f}} . \tag{1.35}$$

Here we encounter a central matrix element of scattering theory

$$T_{q\mathrm{f}q} \equiv \langle \psi^0_{q\mathrm{f}} | V | \Psi^{(+)}_q \rangle \tag{1.36}$$

in terms of which we get

$$A_{q\mathrm{f}}(t) = f_0(q_\mathrm{f}) + \lim_{\varepsilon\to 0} \int dq \exp[\mathrm{i}(E_{q\mathrm{f}} - E_q)t] \frac{T_{q\mathrm{f}q}}{E_q + \mathrm{i}\varepsilon - E_{q\mathrm{f}}} f_0(q) \tag{1.37}$$

and

$$\dot{A}_{q\mathrm{f}}(t) = -\mathrm{i} \int dq \exp[\mathrm{i}(E_{q\mathrm{f}} - E_q)t]\, T_{q\mathrm{f}q} f_0(q)\,. \tag{1.38}$$

Now we are prepared to calculate the transition rate at time t:

$$\begin{aligned}\frac{d}{dt}|A_{q\mathrm{f}}(t)|^2 = 2\,\mathrm{Re}\Big\{ &-\mathrm{i}\int dq \exp[\mathrm{i}(E_{q\mathrm{f}} - E_q)t]\, T_{q\mathrm{f}q} \\ &\times f_0(q) f_0^*(q_\mathrm{f}) - \mathrm{i}\lim_{\varepsilon\to 0}\int dq \exp[\mathrm{i}(E_{q\mathrm{f}} - E_q)t] \\ &\times T_{q\mathrm{f}q} f_0(q) \int dq' \exp[-\mathrm{i}(E_{q\mathrm{f}} - E_{q'})t] \frac{T^*_{q\mathrm{f}q'}}{E_{q'} - \mathrm{i}\varepsilon - E_{q\mathrm{f}}} f_0^*(q')\Big\}\,.\end{aligned} \tag{1.39}$$

We have to expect that it vanishes for large times t. For large times $|\Psi(t)\rangle$ describes the state when the particle has left the domain D and propagates again freely. Therefore the overlap $A_{q\mathrm{f}}(t) = \langle \psi^0_{q\mathrm{f}}(t) \mid \Psi(t)\rangle$ has to be time independent, since the two states belong to the same (free) Schrödinger equation. Indeed using the relation

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\left(\frac{\mathrm{e}^{-\mathrm{i}xt}}{x + \mathrm{i}\varepsilon}\right) = -2\pi\mathrm{i}\,\delta(x) \tag{1.40}$$

we get from (1.37)

$$\begin{aligned}\lim_{t\to\infty} A_{q\mathrm{f}}(t) &= f_0(q_\mathrm{f}) - 2\pi\mathrm{i}\int dq\, \delta(E_{q\mathrm{f}} - E_q)\, T_{q\mathrm{f}q} f_0(q) \\ &= \int dq\, [\delta^3(q - q_\mathrm{f}) - 2\pi\mathrm{i}\,\delta(E_{q\mathrm{f}} - E_q)\, T_{q\mathrm{f}q}] f_0(q) \\ &\equiv \int dq\, S_{q\mathrm{f}q} f_0(q)\,.\end{aligned} \tag{1.41}$$

Clearly the quantity $S_{q\mathrm{f}q}$ is the probability amplitude for scattering from q to q_f and is called the S-matrix element. We shall say more about S in Sect. 1.5.

The probability $|A_{q\mathrm{f}}(t)|^2$ therefore approaches a time independent limit for $t\to\infty$ and its time derivative has to vanish. Mathematically this can also be seen directly from (1.39) using basic properties of Fourier transforms. Given this fact, how does a nonzero cross section arise? The cross section is the ratio of the transition rate to the incoming flux and we will now show that this ratio, as we go towards a stationary limit, will be nonzero. How do we approach the stationary situation in the initial state? The initial wave packet is given in (1.3). Normalized to 1 it describes the motion of one particle. This is reflected in the momentum distribution which sums up to 1:

$$\int dq\, |f_0(q)|^2 = 1\,. \tag{1.42}$$

We introduced in (1.5a) as an example a Gaussian momentum distribution. In that example a decreasing value b will confine the momenta contained in the wave packet more and more to the neighbourhood of q_i. However because of the normalization condition (1.42) $f_0(q)$ cannot tend towards a δ-function. The normalization condition for a sequence of functions defining the δ-function is

$$\int f_\delta(q)\, dq = 1\,, \tag{1.43}$$

which in the Gaussian form leads to

$$f_\delta(q) = \frac{1}{b^3}\,\frac{1}{(\pi)^{3/2}}\,e^{-(q-q_i)^2/b^2}\,. \tag{1.44}$$

Note the different powers in b occurring in (1.5a) and (1.44). We can write

$$f_0(q) = b^{3/2}(2\pi)^{3/4} f_\delta(q) \tag{1.45}$$

and the particle density in the Gaussian wave packet is expressed as

$$|\psi_0(x,t)|^2 = b^3(2\pi)^{3/2}\left|\int dq\, \psi_q^0(x)\,e^{-iE_q t} f_\delta(q)\right|^2\,. \tag{1.46}$$

In the limit $b \to 0$ the property $f_\delta(q) \to \delta^3(q-q_i)$ reduces the integral to the plane wave state $\psi_{q_i}^0 \exp(-iE_{q_i}t)$ which has the constant particle density $(2\pi)^{-3}$. The factor $b^3(2\pi)^{3/2}$ therefore tells us by how much the probability to find the particle in a unit volume for a spreading wave packet is reduced in comparison to the constant probability of a plane wave state. As a consequence, the probability that the incoming particle hits the target of finite dimension and scatters into a final momentum state, described by $|A_{q_f}(t)|^2$, has to be expected to be reduced by the same factor. Indeed this is the case because of (1.45) and the quadratic dependence of $|A|^2$ on $f_0(q)$. The same is then true for the transition rate.

Now this rate, which decreases like b^3 for sharper and sharper energies, has to be divided by the incoming flux. The flux however, being of the form density × velocity, will also carry the factor b^3 in comparison to the constant flux j_0 belonging to a plane wave state. Indeed ($\overleftrightarrow{\nabla} \equiv \vec{\nabla} - \overleftarrow{\nabla}$)

$$j = \frac{1}{2im}(\psi_0^* \overleftrightarrow{\nabla} \psi_0) \tag{1.47}$$

and for $b \to 0$ we get

$$|j| \to b^3 (2\pi)^{3/2} j_0 \quad \text{with} \tag{1.48}$$

$$j_0 = \frac{|q_i|}{m} \frac{1}{(2\pi)^3} . \tag{1.49}$$

Therefore in the ratio between $(d/dt)\,|A_{q_f}(t)|^2$ and $|j|$ the factor $b^3(2\pi)^{3/2}$ cancels and the stationary limit $b \to 0$ can be carried through. Thus instead of $(d/dt)\,|A|^2$ we regard $b^{-3}(d/dt)\,|A|^2$. Using (1.45) we derive from (1.39)

$$\begin{aligned} &\lim_{b\to 0}\left(\frac{1}{b^3(2\pi)^{3/2}}\,\frac{d}{dt}|A_{q_f}(t)|^2\right) \\ &= 2\,\mathrm{Re}\left\{-\mathrm{i}\,\delta^3(q_f - q_i)\,T_{q_f q_i} - \mathrm{i}\lim_{\varepsilon\to 0}|T_{q_f q_i}|^2 \frac{1}{E_{q_i} - \mathrm{i}\varepsilon - E_{q_f}}\right\} \\ &= 2\,\mathrm{Im}\{T_{q_f q_i}\,\delta^3(q_f - q_i) + 2\pi\,\delta(E_{q_f} - E_{q_i})\,|T_{q_f q_i}|^2\}. \end{aligned} \tag{1.50}$$

The first term results from the interference of the initial wave packet, the beam, with the scattered part of the wave function and is present only in the forward direction.

Let us now regard the scattering events which have a momentum different from the initial one. This is described by the second part, which moreover exhibits energy conservation, a property obviously expected in potential scattering. Now depending on the experimental set up we can calculate the number of events occurring per unit time. In potential scattering the most detailed observable is the number of particles scattered per unit time into a solid angle $d\hat{q}_f$ and into a small momentum interval Δq_f. Assuming constancy of $T_{q_f q_i}$ in these intervals that number is [up to the factor $(2\pi)^{3/2} b^3$, which will be cancelled by $|j|$]

$$dN = |T_{q_f q_i}|^2\, d\hat{q}_f \int_{\Delta q_f} dq_f\, q_f^2\, 2\pi\,\delta(E_{q_f} - E_{q_i}) = 2\pi m\, |q_i|\, |T_{q_f q_i}|^2\, d\hat{q}_f . \tag{1.51}$$

Then the differential cross section

$$d\sigma \equiv \frac{dN(2\pi)^{3/2} b^3}{|j|} = \frac{dN}{j_0} \tag{1.52}$$

turns out to be

$$\frac{d\sigma}{d\hat{q}_f} = (2\pi)^4 m^2 |T_{q_f q_i}|^2 . \tag{1.53}$$