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Vitali D. Milman
Gideon Schechtman (Eds.)

Geometric Aspects of Functional Analysis

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Preface

Since the mid-1980's the following volumes containing collections of papers reflecting the activity of the Israel Seminar in Geometric Aspects of Functional Analysis appeared:

1983-1984	Published privately by Tel Aviv University
1985-1986	Springer Lecture Notes, Vol. 1267
1986-1987	Springer Lecture Notes, Vol. 1317
1987-1988	Springer Lecture Notes, Vol. 1376
1989-1990	Springer Lecture Notes, Vol. 1469
1992-1994	Operator Theory: Advances and Applications, Vol. 77, Birkhauser
1994-1996	MSRI Publications, Vol. 34, Cambridge University Press
1996-2000	Springer Lecture Notes, Vol. 1745
2001-2002	Springer Lecture Notes, Vol. 1807
2002-2003	Springer Lecture Notes, Vol. 1850.

Of these, the first six were edited by Lindenstrauss and Milman, the seventh by Ball and Milman and the last three by the two of us.

As in the previous volumes, the current one reflects general trends of the Theory. Most of the papers deal with different aspects of Asymptotic Geometric Analysis, ranging from classical topics in the geometry of convex bodies, to inequalities involving volumes of such bodies or, more generally, log-concave measures, to the study of sections or projections of convex bodies. In many of the papers Probability Theory plays an important role; in some, limit laws for measures associated with convex bodies, resembling Central Limit Theorems, are derived and in others, probabilistic tools are used extensively. There are also papers on related subjects, including a survey on the behavior of the largest eigenvalue of random matrices and some topics in Number Theory.

All the papers here are original research papers (and one invited expository paper) and were subject to the usual standards of refereeing.

As in previous proceedings of the GAFA Seminar, we also list here all the talks given in the seminar as well as talks in related workshops and conferences. We believe this gives a sense of the main directions of research in our area.

We are grateful to Diana Yellin for taking excellent care of the typesetting aspects of this volume.

Vitali Milman
Gideon Schechtman

Contents

Theory of Valuations on Manifolds, IV. New Properties of the Multiplicative Structure

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Summary. This is the fourth part in the series of articles [A4], [A5], [AF] (see also [A3]) where the theory of valuations on manifolds is developed. In this part it is shown that the filtration on valuations is compatible with the product. Then it is proved that the Euler–Verdier involution on smooth valuations is an automorphism of the algebra of valuations. Then an integration functional on valuations with compact support is introduced, and a property of selfduality of valuations is proved. Next a space of generalized valuations is defined, and some basic properties of it are proved. Finally a canonical imbedding of the space of constructible functions on a real analytic manifold into the space of generalized valuations is constructed, and various structures on valuations are compared with known structures on constructible functions.

Contents

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0 Introduction

In convexity there are many geometrically interesting and well known examples of valuations on convex sets: Lebesgue measure, the Euler characteristic, the surface area, mixed volumes, the affine surface area. For a description of older classical developments on this subject we refer to the surveys [MS], [M2]. For the general background on convexity we refer to the book [S].

Approximately during the last decade there was a significant progress in this classical subject which has led to new classification results of various classes of valuations, to discovery of new structures on them. This progress has shed a new light on the notion of valuation which allowed to generalize it to more general setting of valuations on manifolds and on not necessarily convex sets (which do not make sense on a general manifold). On the other hand author's feeling is that the notion of valuation equips smooth manifolds with a new general structure. The development of the theory of valuations on manifolds was started in three previous parts of the series of articles: [A4], [A5] by the author and [AF] by J. Fu and the author. This article in the forth part in this series.

In [A5] the notion of smooth valuation on a smooth manifold was introduced. Roughly speaking a smooth valuation can be thought as a finitely additive $\mathbb{C}$-valued measure on a class of nice subsets; this measure is requested to satisfy some additional assumptions of continuity (or rather smoothness) in some sense. The basic examples of smooth valuations on a general manifold X are smooth measures on X and the Euler characteristic. Moreover the well known intrinsic volumes of convex sets can be generalized to provide examples of smooth valuations on an arbitrary *Riemannian* manifold; these valuations are known as Lipschitz-Killing curvatures.

Let X be a smooth manifold of dimension n. The space of smooth valuations on X is denoted by $V^\infty(X)$. It has a canonical linear topology with respect to which it becomes a Fréchet space.

The space $V^\infty(X)$ carries a canonical multiplicative structure. This structure seems to be of particular interest and importance. When X is an affine space it was defined in [A4] (in even more specific situation of valuations polynomial with respect to translations it was defined in [A3]). For a general manifold X the multiplicative structure was defined in [AF]. The construction in [AF] uses the affine case [A4] and additional tools from the geometric measure theory.

It was shown in [AF] that the product $V^\infty(X) \times V^\infty(X) \to V^\infty(X)$ is a continuous map, and $V^\infty(X)$ becomes a commutative associative algebra with the unit (which is the Euler characteristic). The goal of this article is to study further properties of the multiplicative structure and apply one of them (which we call the Selfduality property) to introduce a new class of generalized valuations.

In [A5] a filtration of $V^\infty(X)$

$$V^\infty(X) = W_0 \supset W_1 \supset \cdots \supset W_n \tag{0.1.1}$$

by closed subspaces was introduced. The first main result of this article (Theorem 3.1.1) says that this filtration is compatible with the product, namely $W_i \cdot W_j \subset W_{i+j}$ (where $W_k = 0$ for $k > n$).

In [A5] the author has introduced a continuous involution $\sigma\colon V^\infty(X) \to V^\infty(X)$ called the Euler–Verdier involution. The second main result of this article says that σ is an algebra automorphism (Theorem 4.1.4).

Let us denote by $V_c^\infty(X)$ the space of compactly supported smooth valuations. Next we introduce in this article the integration functional $\int\colon V_c^\infty(X) \to \mathbb{C}$. Slightly oversimplifying, it is defined by $[\phi \mapsto \phi(X)]$. The third main result is as follows.

Theorem 0.1.1. *Consider the bilinear form*

$$V^\infty(X) \times V_c^\infty(X) \to \mathbb{C}$$

given by $(\phi, \psi) \mapsto \int \phi \cdot \psi$.

This bilinear form is a perfect pairing. More precisely the induced map

$$V^\infty(X) \to \left(V_c^\infty(X)\right)^*$$

is injective and has a dense image with respect to the weak topology on $(V_c^\infty(X))^*$.

This is Theorem 6.1.1 in the paper. Its proof uses the Irreducibility Theorem from [A2] in full generality. Roughly Theorem 0.1.1 can be interpreted as a selfduality property of the space of valuations (at least when the manifold X is compact).

Let us denote $V^{-\infty}(X) := (V_c^\infty(X))^*$. We call $V^{-\infty}(X)$ the space of generalized valuations. We show (Proposition 7.1.3) that $V^{-\infty}(X)$ has a canonical structure of $V^\infty(X)$-module.

In [A5] it was shown that the assignment to any open subset $U \subset X$

$$U \mapsto V^\infty(U)$$

with the natural restriction maps is a sheaf denoted by $\mathcal{V}_X^\infty$. Here we show that

$$U \mapsto V^{-\infty}(U)$$

with the natural restriction maps is also a sheaf which we denote by $\mathcal{V}_X^{-\infty}$. Moreover $\mathcal{V}_X^{-\infty}$ is a sheaf of $\mathcal{V}_X^\infty$-modules (Proposition 7.2.4).

Remind that by [A5] the last term W_n of the filtration (0.1.1) coincides with the space $C^\infty(X, |\omega_X|)$ of smooth densities on X (where $|\omega_X|$ denotes the line bundle of densities on X), and $V^\infty(X)/W_1$ is canonically isomorphic to the space of smooth functions $C^\infty(X)$. In Subsection 7.3 of this article we extend the filtration $\{W_\bullet\}$ to generalized valuations by taking the closure of W_i in the weak topology on $V^{-\infty}(X)$:

$$V^{-\infty}(X) = W_0(V^{-\infty}(X)) \supset W_1(V^{-\infty}(X)) \supset \cdots \supset W_n(V^{-\infty}(X)).$$

We show that $W_n(V^{-\infty}(X))$ is equal to the space $C^{-\infty}(X, |\omega_X|)$ of generalized densities on X (Proposition 7.3.5). It is also shown that $V^{-\infty}(X)/W_1(V^{-\infty}(X))$ is canonically isomorphic to the space $C^{-\infty}(X)$ of generalized valuations on X (Proposition 7.3.6).

The Euler–Verdier involution is extended by continuity in the weak topology to the space of generalized valuations (Subsection 7.4). Also the integration functional extends (uniquely) by continuity in an appropriate topology to generalized valuations with compact support (Subsection 7.4).

In Section 8 we consider valuations on a real analytic manifold X. On such a manifold one has the algebra of constructible functions $\mathcal{F}(X)$ which is a quite well known object (see [KS], Ch. 9). We construct a canonical imbedding of the space $\mathcal{F}(X)$ to the space of generalized valuations $V^{-\infty}(X)$ as a dense subspace. It turns out to be possible to interpret some properties of valuations in more familiar terms of constructible functions. Thus we show that the canonical filtration on $V^{-\infty}(X)$ induces on $\mathcal{F}(X)$ the filtration by codimension of support (Proposition 8.2.2). The restriction of the integration functional to the space of compactly supported constructible functions coincides with the well known functional of integration with respect to the

Euler characteristic (Proposition 8.3.1). The restriction of the Euler–Verdier involution on $V^{-\infty}(X)$ to $\mathcal{F}(X)$ coincides (up to a sign) with the well known Verdier duality operator (Proposition 8.4.1).

Acknowledgement. I express my gratitude J. Bernstein for numerous stimulating discussions. I thank V.D. Milman for his attention to this work. I thank A. Bernig for sharing with me the recent preprint [BB], J. Fu for very helpful explanations on the geometric measure theory, P.D. Milman for useful correspondences regarding subanalytic sets, and P. Schapira for useful discussions on constructible sheaves and functions.

1 Background

In this section we fix some notation and remind various known facts. This section does not contain new results.

In Subsection 1.1 we fix some notation and remind the notions of normal and characteristic cycles of *convex* sets. In Subsection 1.2 we review basic facts on subanalytic sets. Subsection 1.3 collects facts on normal and characteristic cycles. In Subsection 1.4 we review some notions on valuations on manifolds following mostly [A4], [A5], [AF]. Subsection 1.5 is also on valuations, and it reviews the canonical filtration on valuations following [A5].

1.1 Notation

Let V be a finite dimensional real vector space.

- Let $\mathcal{K}(V)$ denote the family of convex compact subsets of V.
- Let $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) denote the set of non-negative (resp. positive) real numbers.
- For a manifold X let us denote by $|\omega_X|$ the line bundle of densities over X.
- For a smooth manifold X let $\mathcal{P}(X)$ denote the family of all simple subpolyhedra of X. (Namely $P \in \mathcal{P}(X)$ iff P is a compact subset of X locally diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}$ for some $0 \leq k \leq n$. For a precise definition see [A5], Subsection 2.1.)
- We denote by $\mathbb{P}_+(V)$ the *oriented projectivization* of V. Namely $\mathbb{P}_+(V)$ is the manifold of oriented lines in V passing through the origin.
- For a vector bundle E over a manifold X let us denote by $\mathbb{P}_+(E)$ the bundle over X whose fiber over any point $x \in X$ is equal to $\mathbb{P}_+(E_x)$ (where E_x denotes the fiber of E over x).
- For a convex compact set $A \in \mathcal{K}(V)$ let us denote by h_A the *supporting functional* of A, $h_A \colon V^* \to \mathbb{R}$. It is defined by

$$h_A(y) := \sup\{y(x)|x \in A\}.$$

- Let L denote the (real) line bundle over $\mathbb{P}_+(V^*)$ such that its fiber over an oriented line $l \in \mathbb{P}_+(V^*)$ is equal to the dual line l^*.
- For a smooth vector bundle E over a manifold X and k being a non-negative integer or infinity, let us denote by $C^k(X,E)$ the space of C^k-smooth sections of E. We denote by $C^k_c(X,E)$ the space of C^k-smooth sections with compact support. Let us denote by $C^{-\infty}(X,E)$ the space of generalized sections of E which is equal by definition to the dual space $(C^\infty_c(X, E^* \otimes |\omega_X|))^*$. We have the canonical imbedding $C^k(X,E) \hookrightarrow C^{-\infty}(X,E)$ (see e.g. [GuS], Ch. VI §1).

Let $K \in \mathcal{K}(V)$. Let $x \in K$.

Definition 1.1.1. *A tangent cone to K at x is the set denoted by T_xK which is equal to the closure of the set $\{y \in V | \exists \varepsilon > 0\ x + \varepsilon y \in K\}$.*

It is easy to see that T_xK is a closed convex cone.

Definition 1.1.2. *A normal cone to K at x is the set*

$$(T_xK)^o := \big\{y \in V^* |\ y(x) \geq 0 \forall x \in T_xK\big\}.$$

Thus $(T_xK)^o$ is also a closed convex cone.

Definition 1.1.3. *Let $K \in \mathcal{K}(V)$. The characteristic cycle of K is the set*

$$CC(K) := \cup_{x \in K} (T_xK)^o.$$

It is easy to see that $CC(K)$ is a closed n-dimensional subset of $T^*V = V \times V^*$ invariant with respect to the multiplication by non-negative numbers acting on the second factor. For some references on the characteristic and normal cycles of various sets see Remark 1.3.1 below.

1.2 Subanalytic Sets

In this subsection we review some basic facts from the theory of subanalytic sets of Hironaka. For more information see [Hi1], [Hi2], [H1], [H2], [BiM], [T], and §8.2 of [KS]. Let X be a real analytic manifold.

Definition 1.2.1. *Let Z be a subset of the manifold X. Z is called subanalytic at a point $x \in X$ if there exists an open neighborhood U of x, compact real analytic manifolds Y^i_j, $i = 1, 2$, $j = 1, \dots, N$, and real analytic maps*

$$f^i_j : Y^i_j \to X$$

such that

$$Z \cap U = U \cap \cup_{j=1}^N (f^1_j(Y^1_j) \backslash f^2_j(Y^2_j)).$$

Z is called subanalytic in X if Z is subanalytic at every point of X.

Proposition 1.2.2. (i) *Let Z be a subanalytic subset of the manifold X. Then the closure and the interior of Z are subanalytic subsets.*
(ii) *The connected components of a subanalytic set are locally finite and subanalytic.*
(iii) *Let Z_1 and Z_2 be subanalytic subsets of the manifold X. Then $Z_1 \cup Z_2$, $Z_1 \cap Z_2$, and $Z_1 \backslash Z_2$ are subanalytic.*

Definition 1.2.3. *Let Z be a subanalytic subset of the manifold X. A point $x \in Z$ is called regular if there exists an open neighborhood U of x in X such that $U \cap Z$ is a submanifold of X.*

The set of regular points is denoted by Z_{reg}. Define the set of *singular* points by $Z_{\text{sing}} := Z \backslash Z_{\text{reg}}$.

Proposition 1.2.4. *The sets Z_{reg} and Z_{sing} are subanalytic, and $Z \subset \bar{Z}_{\text{reg}}$.*

If $x \in Z_{\text{reg}}$ then the dimension of Z at x is well defined; it is denoted by $\dim_x Z$. Define

$$\dim Z := \sup_{x \in Z_{\text{reg}}} \dim_x(Z).$$

Clearly $\dim Z \leq \dim X$.

Proposition 1.2.5. *Let $Z \subset X$ be a subanalytic subset. Then*

(i) $\dim(Z \backslash Z_{\text{reg}}) < \dim Z$;
(ii) $\dim(\bar{Z} \backslash Z) < \dim Z$.

Definition 1.2.6 ([KS], §9.7). *An integer valued function $f\colon X \to \mathbb{Z}$ is called constructible if*

1) *for any $m \in \mathbb{Z}$ the set $f^{-1}(m)$ is subanalytic;*
2) *the family of sets $\{f^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite.*

Clearly the set of constructible $\mathbb{Z}$-valued functions is a ring with pointwise multiplication. As in [KS] we denote this ring by $CF(X)$. Define

$$\mathcal{F} := CF(X) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{1.2.1}$$

Thus $\mathcal{F}$ is a subalgebra of the $\mathbb{C}$-algebra of complex valued functions on X. In the rest of the article the elements of $\mathcal{F}$ will be called *constructible functions.*

Let $\mathcal{F}_c(X)$ denote the subspace of $\mathcal{F}(X)$ of *compactly supported* constructible functions. Clearly $\mathcal{F}_c(X)$ is a subalgebra of $\mathcal{F}(X)$ (without unit if X is non-compact).

For a subset $P \subset X$ let us denote by $1\!\!1_P$ the indicator function of P, namely

$$1\!\!1_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P. \end{cases}$$

Proposition 1.2.7. (i) *Any function $f \in \mathcal{F}(X)$ can be presented locally as finite linear combination of functions of the form $1\!\!1_Q$ where Q is a closed subanalytic subset.*
(ii) *Any function $f \in \mathcal{F}_c(X)$ can be presented as finite linear combination of functions of the form $1\!\!1_Q$ where Q is a compact subanalytic subset.*

Proof. Both statements are proved similarly. Let prove say the second one. Let $f \in \mathcal{F}_c(X)$. We prove the statement by the induction on $\dim(\operatorname{supp} f)$ (note that $\operatorname{supp} f$ is a subanalytic subset). If $\dim(\operatorname{supp} f) = 0$ then there is nothing to prove. Let us assume that we have proven the results for all constructible functions with the dimension of support strictly less than k. Let us prove it for k. Clearly f is a finite linear combination of functions of the form $1\!\!1_Q$ where Q is relatively compact subanalytic subset with $\dim Q \leq k$. But

$$1\!\!1_Q = 1\!\!1_{\bar{Q}} - 1\!\!1_{\bar{Q}\backslash Q}.$$

By Proposition 1.2.2 the set $\bar{Q}\backslash Q$ is subanalytic, and by Proposition 1.2.5(ii) $\dim(\bar{Q}\backslash Q) < k$. The induction assumption implies the result. □

1.3 Characteristic and Normal Cycles

In Subsection 1.1 we have reminded the notion of characteristic cycle of convex compact sets. In this subsection we remind the notion of characteristic cycle and very similar notion of normal cycles of sets either from the class $\mathcal{P}(X)$ on a smooth manifold X, or the class of subanalytic subsets of a real analytic manifold X (in fact in the real analytic situation these notions will be discussed more generally for constructible functions on X following [KS]). The notions of characteristic and normal cycles of various classes of sets coincide on the pairwise intersections of these classes.

Remark 1.3.1. The notion of the characteristic cycle is not new. First an almost equivalent notion of normal cycle (see below) was introduced by Wintgen [W], and then studied further by Zähle [Z] by the tools of geometric measure theory. Characteristic cycles of subanalytic sets of real analytic manifolds were introduced by J. Fu [F2] using the tools of geometric measure theory and independently by Kashiwara (see [KS], Chapter 9) using the tools of the sheaf theory. J. Fu's article [F2] develops a more general approach to define the normal cycle for more general sets than subanalytic or convex ones (see Theorem 3.2 in [F2]). Applications of the method of normal cycles to integral geometry can be found in [F1].

For simplicity of the exposition, in the rest of this subsection we will assume that the manifold X is oriented. Then characteristic (resp. normal) cycle is a cycle in T^*X (resp. $\mathbb{P}_+(T^*X)$). Nevertheless the characteristic and normal cycles can be defined on non-oriented (even non-orientable) manifolds; then they are cycles taking values is the local system p^*o where o is the orientation