Lecture Notes in Mathematics

Karl Friedrich Siburg

The Principle of Least Action in Geometry and Dynamics

1844

$$\alpha(h) = \min\left\{ \int L \,\mathrm{d}\mu \mid \rho(\mu) = h \right\}$$

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Preface

The motion of classical mechanical systems is determined by Hamilton's differential equations:

$$\begin{cases} \dot{x}(t) = \partial_y H(x(t), y(t)) \\ \dot{y}(t) = -\partial_x H(x(t), y(t)) \end{cases}$$

For instance, if we consider the motion of n particles in a potential field, the Hamiltonian function

$$H = \frac{1}{2} \sum_{i=1}^{n} y_i^2 - V(x_1, \ldots, x_n)$$

is the sum of kinetic and potential energy; this is just another formulation of Newton's Second Law.

A distinguished class of Hamiltonians on a cotangent bundle T^*X consists of those satisfying the Legendre condition. These Hamiltonians are obtained from Lagrangian systems on the configuration space X, with coordinates $(x, \dot{x})$ = (*space, velocity*), by introducing the new coordinates (x, y) = (*space, momentum*) on its phase space T^*X. Analytically, the Legendre condition corresponds to the convexity of H with respect to the fiber variable y. The Hamiltonian gives the energy value along a solution (which is preserved for time–independent systems) whereas the Lagrangian describes the action. Hamilton's equations are equivalent to the Euler–Lagrange equations for the Lagrangian:

$$\frac{\mathrm{d}}{\mathrm{d}t} \partial_{\dot{x}} L(x(t), \dot{x}(t)) = \partial_x L(x(t), \dot{x}(t)).$$

These equations express the variational character of solutions of the Lagrangian system. A curve $x : [t_0, t_1] \to \mathbb{R}^n$ is a Euler–Lagrange trajectory if, and only if, the first variation of the action integral, with end points held fixed, vanishes:

$$\delta \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, \mathrm{d}t \Big|_{x(t_0)}^{x(t_1)} = 0.$$

In other words, solutions extremize the action with fixed end points on each finite time interval.

This is not quite what one usually remembers from school[1], namely that solutions should *minimize* the action. The crucial point here is that the minimizing property holds only for short times. For instance, when looking at geodesics on the round sphere, the movement along a great circle ceases to be the shortest connection as soon as one comes across the antipodal point.

However, under certain circumstances there may well be action minimizing trajectories. The investigation of these minimal objects is one of the central topics of the present work. In fact, they do not always exist as genuine solutions, but they do so as invariant measures. This is the outcome of a theory by Mather and Mañé which generalizes Aubry–Mather theory from one to more degrees of freedom. In particular, there exist action minimizing measures with any prescribed "asymptotic direction" (described by a homological rotation vector). Associating to each such rotation vector the action of a minimal measure, we obtain the *minimal action* functional

$$\alpha : H_1(X, \mathbb{R}) \to \mathbb{R}.$$

By construction, the minimal action does not describe the full dynamics but concentrates on a very special part of it. The fundamental question is how much information about the original system is contained in the minimal action?

The first two chapters of this book provide the necessary background on Aubry–Mather and Mather–Mañé theories. In the following chapters, we investigate the minimal action in four different settings:

1. convex billiards
2. fixed points and invariant tori
3. Hofer's geometry
4. symplectic geometry.

We will see that the minimal action plays an important role in all four situations, underlining the significance of that particular variational principle.

1. Convex billards. Can one hear the shape of a drum? This was Kac' pointed formulation of the inverse spectral problem: is a manifold uniquely determined by its Laplace spectrum? We do know now that this is not true in full generality; for the class of smooth convex domains in the plane, however, this problem is still open.

We ask a somewhat weaker question for the length spectrum (i.e., the set of lengths of closed geodesics) rather than the Laplace spectrum, which is closely related to the previous one: how much geometry of a convex domain is determined by its length spectrum? The crucial observation is that one can consider this geometric problem from a more dynamical viewpoint. Namely,

[1] depending on the school, of course...

following a geodesic inside a convex domain that gets reflected at the boundary, is equivalent to iterating the so–called billiard ball map. The latter is a monotone twist map for which the minimal action is defined.

The main results from Chapter 3 can be summarized as follows.

Theorem 1. *For planar convex domains, the minimal action is invariant under continuous deformations of the domain that preserve the length spectrum.*

In particular, every geometric quantity that can be written in terms of the minimal action is automatically a length spectrum invariant.

In fact, the minimal action is a complete invariant and puts all previously known ones (e.g., those constructed in [2, 19, 63, 87]) into a common framework.

2. Fixed points and invariant tori. We consider a symplectic diffeomorphism in a neighbourhood of an elliptic fixed point in $\mathbb{R}^2$. If the fixed point is of "general" type, the symplectic character of the map makes it possible (under certain restrictions) to find new symplectic coordinates in which the map takes a particularly simple form, the so–called Birkhoff normal form. The coefficients of this normal form, called Birkhoff invariants, are symplectically invariant.

The Birkhoff normal form describes an asymptotic approximation, in the sense that it coincides with the original map only up to a term that vanishes asymptotically when one approaches the fixed point. In general, it does not give any information about the dynamics away from the fixed point.

The main result in this context introduces the minimal action as a symplectically invariant function that contains the Birkhoff normal form, but also reflects part of the dynamics near the fixed point.

Theorem 2. *Associated to an area–preserving map near a general elliptic fixed point there is the minimal action α, which is symplectically invariant.*

It is a local *invariant, i.e., it contains information about the dynamics near the fixed point. Moreover, the Taylor coefficients of the convex conjugate α^* are the Birkhoff invariants.*

Area–preserving maps near a fixed point occur as Poincaré maps of closed characteristics of three–dimensional contact flows. A particular example is given by the geodesic flow on a two–dimensional Riemannian manifold. In this case, the minimal action is determined by the length spectrum of the surface, and we obtain the following result.

Theorem 3. *Associated to a general elliptic closed geodesic on a two–dimensional Riemannian manifold there is the germ of the minimal action, which is a length spectrum invariant under continuous deformations of the Riemannian metric.*

The minimal action carries information about the geodesic flow near the closed geodesic; in particular, it determines its C^0–integrability.

In higher dimensions, we consider a symplectic diffeomorphism ϕ in a neighbourhood of an invariant torus Λ. If we assume that the dynamics on Λ satisfy a certain non–resonance condition, one can transform ϕ into Birkhoff normal form again. If this normal form is positive definite the map ϕ determines the germ of the minimal action α, and we will show again that the minimal action contains the Birkhoff invariants as Taylor coefficients of α^*.

3. Hofer's geometry. Whereas the first three settings had many features in common, the viewpoint here is quite different. Instead of looking at a single Hamiltonian system, we investigate all Hamiltonian systems on a symplectic manifold (M, ω) at once, collected in the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$. It is one of the cornerstones of symplectic topology that this group carries a bi–invariant Finsler metric d, usually called Hofer metric, which is constructed as follows.

Think of $\mathrm{Ham}(M, \omega)$ as infinite–dimensional Lie group whose Lie algebra consists of all smooth, compactly supported functions $H : M \to \mathbb{R}$ with mean value zero. Introduce any norm $\|\cdot\|$ on those functions that is invariant under the adjoint action $H \mapsto H \circ \psi^{-1}$. Then the Hofer distance of a diffeomorphism ϕ from the identity is defined as the infimum of the lengths of all paths in $\mathrm{Ham}(M, \omega)$ that connect ϕ to the identity:

$$d(\mathrm{id}, \phi) = \inf \Big\{ \int_0^1 \|H_t\| \,\mathrm{d}t \mid \varphi_H^1 = \phi \Big\}.$$

The problem is to choose the norm $\|\cdot\|$. The Hamiltonian system is determined by the first derivatives of H, but $\|\mathrm{d}H\|_{C^0}$, for instance, is not invariant under the adjoint action. It turns out that the oscillation norm

$$\|\cdot\| = \mathrm{osc} := \max - \min$$

is the right choice although it seems to have nothing to do with the dynamics. Loosely speaking, the Hofer metric generates a C^{-1}–topology and measures how much energy is needed to generate a given map.

The resulting geometry is far from being understood completely. This is due to the fact that, despite its simple definition, the Hofer distance is very hard to compute. After all, one has to take *all* Hamiltonians into account that generate the same time–1–map. A fundamental question concerns the relation between the Hofer geometry and dynamical properties of a Hamiltonian diffeomorphism: does the dynamical behaviour influence the Hofer geometry and, vice versa, can one infer knowledge about the dynamics from Hofer's geometry? Only little is known in this direction.

In Chap. 5, we take up this question for Hamiltonians on the cotangent bundle $T^*\mathbb{T}^n$ satisfying a Legendre condition. This leads to convex Lagrangians on $T\mathbb{T}^n$ for which the minimal action α is defined. On the other hand, the Hamiltonians under consideration are unbounded and do not fit into the framework of Hofer's metric. Therefore, we have to restrict them to a compact part of $T^*\mathbb{T}^n$, e.g., to the unit ball cotangent bundle $B^*\mathbb{T}^n$, but in such a way that we stay in the range of Mather's theory.

Let α denote the minimal action associated to a convex Hamiltonian diffeomorphism on $B^*\mathbb{T}^n$. Our main result in this context shows that the oscillation of α^*, which is nothing but $\alpha(0)$, is a lower bound for the Hofer distance. This establishes a link between Hofer's geometry of convex Hamiltonian mappings and their dynamical behaviour.

Theorem 4. *If $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian then*

$$d(id, \phi) \geq \operatorname{osc} \alpha^* = \alpha(0).$$

4. Symplectic geometry. Consider the cotangent bundle $T^*\mathbb{T}^n$ with its canonical symplectic form $\omega_0 = \mathrm{d}\lambda$. Here, λ is the Liouville 1–form which is $y\,\mathrm{d}x$ in local coordinates (x, y). Suppose $H : T^*\mathbb{T}^n \to \mathbb{R}$ is a convex Hamiltonian. Because H is time–independent the energy is preserved under the corresponding flow, i.e., all trajectories lie on (fiberwise) convex $(2n-1)$–dimensional hypersurfaces $\Sigma = \{H = \text{const.}\}$. Of particular importance in classical mechanics are so–called KAM–tori. i.e., invariant tori carrying quasiperiodic motion. These are graphs over the base manifold $\mathbb{T}^n$, with the additional property that the symplectic form ω_0 vanishes on them; submanifolds with the latter property are called Lagrangian submanifolds.

We want to study symplectic properties of Lagrangian submanifolds on convex hypersurfaces. To do so, we observe that a Lagrangian submanifold possesses a Liouville class a_Λ, induced by the pull-back of the Liouville form λ to Λ. The Liouville class is invariant under Hamiltonian diffeomorphisms, i.e., it belongs to the realm of symplectic geometry. On the other hand, being a graph is certainly *not* a symplectic property. Our starting question in this context is as follows: is it possible to move a Lagrangian submanifold Λ on some convex hypersurface Σ by a Hamiltonian diffeomorphism inside the domain U_Σ bounded by Σ?

In a first part, we will see that, under certain conditions on the dynamics on Λ, it is impossible to move Λ at all; we call this phenomenon *boundary rigidity.* In fact, the Liouville class a_Λ already determines Λ uniquely.

Theorem 5. *Let Λ be a Lagrangian submanifold with conservative dynamics that is contained in a convex hypersurface Σ, and let K be another Lagrangian submanifold inside U_Σ. Then*

$$a_\Lambda = a_K \iff \Lambda = K.$$

What can happen if boundary rigidity fails? Surprisingly, even if it is possible to push Λ partly inside the domain U_Σ, it cannot be done completely. Certain pieces of Λ have to stay put, and we call them *non–removable intersections.* In the case where Σ is some distinguished "critical" level set, these non–removable intersections always contain an invariant subset with specific dynamical behaviour; this subset is the so–called Aubry set from Mather–Mañé theory. This result reveals a hidden link between aspects of symplectic geometry and Mather–Mañé theory in modern dynamical systems.

Finally, we come back to the somewhat annoying fact that the property of being a Lagrangian section is not preserved under Hamiltonian diffeomorphisms. For this, we consider

Theorem 6. *Let U be a (fiberwise) convex subset U of $T^*\mathbb{T}^n$. Then every cohomology class that can be represented as the Liouville class of some Lagrangian submanifold in U, can actually be represented by a Lagrangian* section *contained in U.*

So, from this rather vague point of view at least, Lagrangian sections actually do belong to symplectic geometry.

Furthermore, the above result allows *symplectic* descriptions of seemingly non–symplectic objects: the stable norm from geometric measure theory, and also our favourite, the minimal action.

Theorem 7. *The stable norm of a Riemannian metric g on $\mathbb{T}^n$, and the minimal action of a convex Lagrangian $L : T\mathbb{T}^n \to \mathbb{R}$, both admit a symplectically invariant description.*

This closes the circle for our investigation of the Principle of Least Action in geometry and dynamics.

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Contents

1

Aubry–Mather theory

The Principle of Least Action states that, for sufficiently short times, trajectories of a Lagrangian system minimize the action amongst all paths in configuration space with the same end points. If the time interval becomes larger, however, the Euler–Lagrange equations describe just critical points of the action functional; they may well be saddle points.

In the eighties, Aubry [5] and Mather [64] discovered independently that monotone twist maps on an annulus possess orbits of any given rotation number which minimize the (discrete) action with fixed end points on *all* time intervals. Roughly speaking, the rotation number of a geodesic describes the direction in which the geodesic, lifted to the universal cover, travels. Those minimal orbits turned out to be of crucial importance for a deeper understanding of the complicated orbit structure of monotone twist mappings.

Later, Mather [69] developed a similar theory for Lagrangian systems in higher dimensions. There was, however, an old example by Hedlund [41] of a Riemannian metric on $\mathbb{T}^3$, having only three directions for which minimal geodesics existed. Therefore, Mather's generalization deals with minimal invariant measures instead of minimal orbits.

A different approach was suggested by Mañé [62] who introduced a certain critical energy value at which the dynamics of a Lagrangian systems change. It turned out that this approach essentially contains Mather's theory, but in a more both geometrical and dynamical setting.

We will deal with these generalizations of Aubry–Mather theory to higher dimensions in Chap. 2.

1.1 Monotone twist mappings

Let

$$\mathbb{S}^1 \times (a, b) \subset \mathbb{S}^1 \times \mathbb{R} = T^*\mathbb{S}^1$$

be a plane annulus with $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, where we allow the cases $a = -\infty$ or $b = +\infty$ (or both). Given a diffeomorphism ϕ of $\mathbb{S}^1 \times (a, b)$ we consider a lift $\widetilde{\phi}$

of ϕ to the universal cover $\mathbb{R} \times (a, b)$ of $\mathbb{S}^1 \times (a, b)$ with coordinates x, y. Since ϕ is a diffeomorphism, so is $\widetilde{\phi}$, and we have $\widetilde{\phi}(x + 1, y) = \widetilde{\phi}(x, y) + (1, 0)$. In this section, we will always work with (fixed) lifts for which we drop the tilde again and keep the notation ϕ.

In the case when a or b is finite we assume that ϕ extends continuously to $\mathbb{R} \times [a, b]$ by rotations by some fixed angles:

$$\phi(x, a) = (x + \omega_-, a) \quad \text{and} \quad \phi(x, b) = (x + \omega_+, b). \tag{1.1}$$

The numbers $\omega_\pm$ are unique after we have fixed the lift. For simplicity, we set $\omega_\pm = \pm\infty$ if $a = -\infty$ or $b = \infty$.

Definition 1.1.1. *A monotone* twist map *is a C^1–diffeomorphism*

$$\begin{aligned} \phi : \mathbb{R} \times (a, b) &\to \mathbb{R} \times (a, b) \\ (x_0, y_0) &\mapsto (x_1, y_1) \end{aligned}$$

satisfying $\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$ as well as the following conditions:

1. *ϕ preserves orientation and the boundaries of $\mathbb{R} \times (a, b)$, in the sense that $y_1(x_0, y_0) \to a, b$ as $y_0 \to a, b$;*
2. *if a or b is finite ϕ extends to the boundary by a rotation, i.e., it satisfies (1.1);*
3. *ϕ satisfies a monotone* twist condition

$$\frac{\partial x_1}{\partial y_0} > 0; \tag{1.2}$$

4. *ϕ is exact symplectic; in other words, there is a C^2–function h, called a* generating function *for ϕ, such that*

$$y_1 \, \mathrm{d}x_1 - y_0 \, \mathrm{d}x_0 = \mathrm{d}h(x_0, x_1). \tag{1.3}$$

The interval $(\omega_-, \omega_+) \subset \mathbb{R}$, which can be infinite, is called the twist interval *of ϕ.*

Remark 1.1.2. The twist condition (1.2) states that images of verticals are graphs over the x–axis; see Fig. 1.1. This implies that ϕ can be described in the coordinates x_0, x_1 rather than x_0, y_0. In other words, for every choice of x–coordinates x_0 and x_1 (corresponding to the configuration space), there are unique choices y_0 and y_1 for the y–coordinates (corresponding to the velocities) such that the image of (x_0, y_0) under ϕ is (x_1, y_1).

Remark 1.1.3. A generating function h for a twist map ϕ is defined on the strip

$$\{(\xi, \eta) \in \mathbb{R}^2 \mid \omega_- < \eta - \xi < \omega_+\}$$

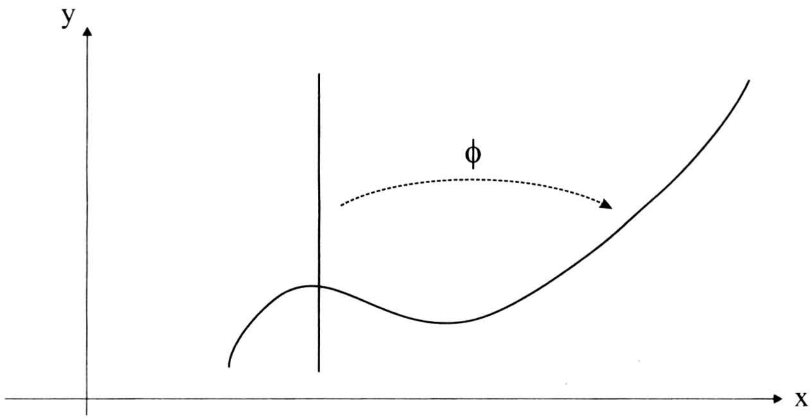

Fig. 1.1. The twist condition

and can be extended continuously to its closure. It is unique up to additive constants. Equation (1.3) is equivalent to the system

$$\begin{cases} \partial_1 h(x_0, x_1) = -y_0 \\ \partial_2 h(x_0, x_1) = y_1 \end{cases} \tag{1.4}$$

Here, the expression ∂_i denotes the partial derivative of a function with respect to the i–th variable. The equivalent of the twist condition (1.2) for a generating function is

$$\partial_1 \partial_2 h < 0. \tag{1.5}$$

Finally, a generating function satisfies the periodicity condition $h(\xi+1, \eta+1) = h(\xi, \eta)$.

Monotone twist maps are not as artificial as they might seem. They appear in a variety of situations, often unexpected and detected only by clever coordinate choices. In the following, we give a few examples. The reader my consult

Example 1.1.4. The simplest example is what is called an *integrable twist map* which, by definition, preserves the radial coordinate[1]. In this case, the property of being area–preserving implies that an integrable twist map is of the following form:

$$\phi(x_0, y_0) = (x_0 + f(y_0), y_0)$$

with $f' > 0$. Then the generating function (up to additive constants) is given by

[1] In the context of integrable Hamiltonian systems, this means that (x, y) are already the angle–action–variables.

$$h = h(x_1 - x_0),$$

with $h' = f^{-1}$; in other words, h is strictly convex.

Example 1.1.5. In some sense the "simplest" non–integrable monotone twist map is the so–called standard map

$$\phi : (x, y) \mapsto \left(x + y + \frac{k}{2\pi} \sin 2\pi x, y + \frac{k}{2\pi} \sin 2\pi x\right)$$

where $k \geq 0$ is a parameter. This map has been the subject of extensive analytical and numerical studies. Famous pictures illustrate the transition from integrability ($k = 0$) to "chaos" ($k \approx 10$).

Example 1.1.6. A particularly interesting class of monotone twist maps comes from planar convex billiards; we will deal with convex billiards in Chap. 3. The investigation of such systems goes back to Birkhoff [15] who introduced them as model case for nonlinear dynamical systems; for a modern survey see [101].

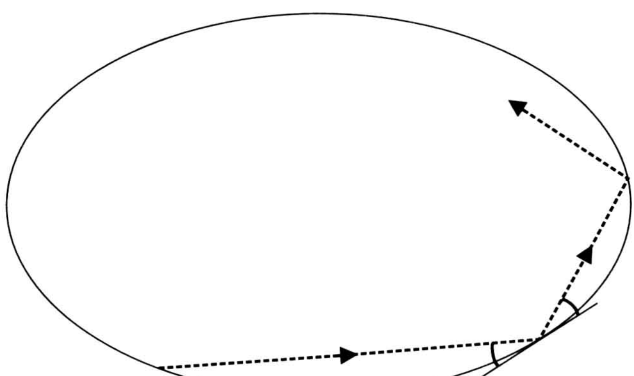

Fig. 1.2. The billard in a strictly convex domain

Given a strictly convex domain Ω in the Euclidean plane with smooth boundary $\partial\Omega$, we play the following game. Let a mass point move freely inside Ω, starting at some initial point on the boundary with some initial direction pointing into Ω. When the "billiard ball" hits the boundary, it gets reflected according to the rule "angle of incidence = angle of reflection"; see Fig. 1.2. The billiard map associates to a pair (point on the boundary, direction), respectively (s, ψ) = (arclength parameter divided by total length, angle with the tangent), the corresponding data when the points hits the boundary again. The lift of this map, which is then defined on $\mathbb{R} \times (0, \pi)$, is not a monotone twist map.

However, elementary geometry shows [101] that the map preserves the 2–form

$$\sin\psi \, \mathrm{d}\psi \wedge \mathrm{d}s = \mathrm{d}(-\cos\psi) \wedge \mathrm{d}s.$$

Hence the billiard map preserves the standard area form $\mathrm{d}x \wedge \mathrm{d}y$ in the new coordinates

$$(x, y) = (s, -\cos\psi) \in \mathbb{R} \times (-1, 1).$$

Moreover, if you increase the angle with the positive tangent to $\partial\Omega$ for the initial direction, the point where you hit $\partial\Omega$ again will move around $\partial\Omega$ in positive direction. This means that

$$\frac{\partial x_1}{\partial y_0} > 0,$$

so the billiard map in the new coordinates does satisfy the monotone twist condition.

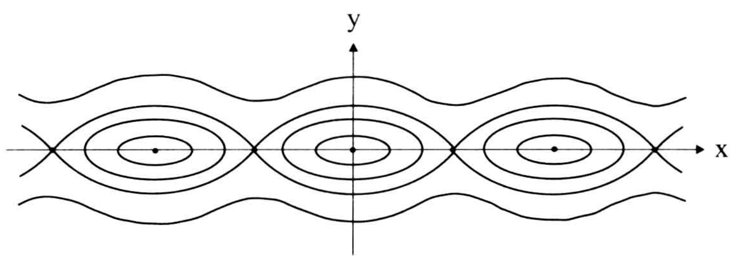

Fig. 1.3. The phase portrait of the mathematical pendulum

Example 1.1.7. Consider a particle moving in a periodic potential on the real line. According to Newton's Second Law, the motion of the particle is determined by the differential equation

$$\ddot{x}(t) = V'(x(t)).$$

This can be written as a Hamiltonian system

$$\begin{cases} \dot{x}(t) = \partial_y H(x(t), y(t)) \\ \dot{y}(t) = -\partial_x H(x(t), y(t)) \end{cases}$$

with the Hamiltonian $H(x, y) = y^2/2 - V(x)$. For small enough $t > 0$, we have

$$\begin{aligned} \frac{\partial x(t; x(0), y(0))}{\partial y(0)} &= \frac{\partial}{\partial y(0)} \int_0^t \dot{x}(\tau; x(0), y(0)) \, d\tau \\ &= \int_0^t \frac{\partial y(\tau; x(0), y(0))}{\partial y(0)} \, d\tau \\ &> 0. \end{aligned}$$

Therefore the time–t–map φ_H^t is a monotone twist map provided t is small. In fact, this holds true not only for Hamiltonians of the form "kinetic energy + potential energy", but for more general Hamiltonians which are fiberwise convex in the second variable (corresponding to the momentum).

A particular case is that of a mathematical pendulum where x is the angle to the vertical and $V'(x) = -\sin 2\pi x$. The phase portrait in $\mathbb{R} \times \mathbb{R}$, see Fig. 1.3, shows two types of invariant curves: closed circles around the stable equilibrium ("librational" circles), and curves homotopic to the real line above and below the separatrices ("rotational" curves).

Note that, by the monotone twist condition, an orbit $((x_i, y_i))_{i\in\mathbb{Z}}$ of a monotone twist map ϕ is completely determined by the sequence $(x_i)_{i\in\mathbb{Z}}$ via the relations

$$y_i = \partial_2 h(x_{i-1}, x_i) = -\partial_1 h(x_i, x_{i+1}).$$

Similarly, an arbitrary sequence $(\xi_i)_{i\in\mathbb{Z}}$ corresponds to an orbit of a monotone twist map ϕ if and only if

$$\partial_2 h(\xi_{i-1}, \xi_i) + \partial_1 h(\xi_i, \xi_{i+1}) = 0 \tag{1.6}$$

for all $i \in \mathbb{Z}$. Thus, on a formal level, orbits of a monotone twist mapping may be regarded as "critical points" of the discrete action "functional"

$$(\xi_i)_{i\in\mathbb{Z}} \mapsto \sum_{i\in\mathbb{Z}} h(\xi_i, \xi_{i+1})$$

on $\mathbb{R}^{\mathbb{Z}}$. This point of view leads to the following notion of minimal orbits.

1.2 Minimal orbits

Let $\phi : (x_0, y_0) \mapsto (x_1, y_1)$ be a monotone twist map with generating function $h(x_0, x_1)$. We have seen above that the ϕ–orbit of a point (x_0, y_0) is completely determined by the sequence (x_i) of the first coordinates. Moreover, an arbitrary sequence (ξ_i) corresponds to an orbit if, and only if, it satisfies the recursive relation (1.6). Loosely speaking, orbits are "critical points" of the action "functional"

$$(\xi_i)_{i\in\mathbb{Z}} \mapsto \sum_{i\in\mathbb{Z}} h(\xi_i, \xi_{i+1}).$$

In this section, we are interested in minima, i.e. in points which minimize the action.

This, of course, makes only sense if we restrict the action of a sequence $(\xi_i)_{i\in\mathbb{Z}}$ to finite parts. In analogy to the classical Principle of Least Action, we define minimal orbits in such a way that they minimize the action with the end points held fixed.