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René L. Schilling, Renming Song, Zoran Vondraček

BERNSTEIN FUNCTIONS

THEORY AND APPLICATIONS

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Theory and Applications

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Preface

Bernstein functions and the important subclass of complete Bernstein functions appear in various fields of mathematics—often with different definitions and under different names. Probabilists, for example, know Bernstein functions as Laplace exponents, and in harmonic analysis they are called negative definite functions. Complete Bernstein functions are used in complex analysis under the name Pick or Nevanlinna functions, while in matrix analysis and operator theory, the name operator monotone function is more common. When studying the positivity of solutions of Volterra integral equations, various types of kernels appear which are related to Bernstein functions. There exists a considerable amount of literature on each of these classes, but only a handful of texts observe the connections between them or use methods from several mathematical disciplines.

This book is about these connections. Although many readers may not be familiar with the name *Bernstein function*, and even fewer will have heard of *complete Bernstein functions*, we are certain that most have come across these families in their own research. Most likely only certain aspects of these classes of functions were important for the problems at hand and they could be solved on an *ad hoc* basis. This explains quite a few of the rediscoveries in the field, but also that many results and examples are scattered throughout the literature; the exceedingly rich structure connecting this material got lost in the process. Our motivation for writing this book was to point out many of these connections and to present the material in a unified way. We hope that our presentation is accessible to researchers and graduate students with different backgrounds. The results as such are mostly known, but our approach and some of the proofs are new: we emphasize the structural analogies between the function classes which we believe is a very good way to approach the topic. Since it is always important to know explicit examples, we took great care to collect many of them in the tables which form the last part of the book.

Completely monotone functions—these are the Laplace transforms of measures on the half-line $[0, \infty)$—and Bernstein functions are intimately connected. The derivative of a Bernstein function is completely monotone; on the other hand, the primitive of a completely monotone function is a Bernstein function if it is positive. This observation leads to an integral representation for Bernstein functions: the Lévy–Khintchine formula on the half-line

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt), \quad \lambda > 0.$$

Although this is familiar territory to a probabilist, this way of deriving the Lévy–Khintchine formula is not the usual one in probability theory. There are many more connections between Bernstein and completely monotone functions. For example, f is a Bernstein function if, and only if, for all completely monotone functions g the composition $g \circ f$ is completely monotone. Since g is a Laplace transform, it is enough to check this for the kernel of the Laplace transform, i.e. the basic completely monotone functions $g(\lambda) = e^{-t\lambda}$, $t > 0$.

A similar connection exists between the Laplace transforms of completely monotone functions, that is, *double Laplace* or *Stieltjes transforms*, and *complete* Bernstein functions. A function f is a complete Bernstein function if, and only if, for each $t > 0$ the composition $(t + f(\lambda))^{-1}$ of the Stieltjes kernel $(t + \lambda)^{-1}$ with f is a Stieltjes function. Note that $(t + \lambda)^{-1}$ is the Laplace transform of $e^{-t\lambda}$ and thus the functions $(t + \lambda)^{-1}$, $t > 0$, are the basic Stieltjes functions. With some effort one can check that complete Bernstein functions are exactly those Bernstein functions where the measure μ in the Lévy–Khintchine formula has a completely monotone density with respect to Lebesgue measure. From there it is possible to get a surprising geometric characterization of these functions: they are non-negative on $(0, \infty)$, have an analytic extension to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$ and preserve upper and lower half-planes. A familiar sight for a classical complex analyst: these are the Nevanlinna functions. One could go on with such connections, delving into continued fractions, continue into interpolation theory and from there to operator monotone functions ...

Let us become a bit more concrete and illustrate our approach with an example. The fractional powers $\lambda \mapsto \lambda^\alpha$, $\lambda > 0$, $0 < \alpha < 1$, are easily among the most prominent (complete) Bernstein functions. Recall that

$$f_\alpha(\lambda) := \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda t})\, t^{-\alpha-1}\, dt. \tag{1}$$

Depending on your mathematical background, there are many different ways to derive and to interpret (1), but we will follow probabilists' custom and call (1) the Lévy–Khintchine representation of the Bernstein function f_α. At this point we do not want to go into details, instead we insist that one should read this formula as an integral representation of f_α with the kernel $(1 - e^{-\lambda t})$ and the measure $c_\alpha\, t^{-\alpha-1}\, dt$.

This brings us to negative powers, and there is another classical representation

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^{\beta-1}\, dt, \quad \beta > 0, \tag{2}$$

showing that $\lambda \mapsto \lambda^{-\beta}$ is a completely monotone function. It is no accident that the reciprocal of the Bernstein function λ^α, $0 < \alpha < 1$, is completely monotone, nor is it an accident that the representing measure $c_\alpha\, t^{-\alpha-1}\, dt$ of λ^α has a completely monotone density. Inserting the representation (2) for $t^{-\alpha-1}$ into (1) and working out the double integral and the constant, leads to the second important formula for the fractional powers,

$$\lambda^\alpha = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{\lambda}{\lambda + t}\, t^{\alpha-1}\, dt. \tag{3}$$

We will call this representation of λ^α the Stieltjes representation. To explain why this is indeed an appropriate name, let us go back to (2) and observe that $t^{\alpha-1}$ is a Laplace transform. This shows that $\lambda^{-\alpha}$, $\alpha > 0$, is a double Laplace or Stieltjes transform. Another non-random coincidence is that

$$\frac{f_\alpha(\lambda)}{\lambda} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{1}{\lambda + t} t^{\alpha-1}\, dt$$

is a Stieltjes transform and so is $\lambda^{-\alpha} = 1/f_\alpha(\lambda)$. This we can see if we replace $t^{\alpha-1}$ by its integral representation (2) and use Fubini's theorem:

$$\frac{1}{f_\alpha(\lambda)} = \lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{1}{\lambda + t} t^{-\alpha}\, dt. \tag{4}$$

It is also easy to see that the fractional powers $\lambda \mapsto \lambda^\alpha = \exp(\alpha \log \lambda)$ extend analytically to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$. Moreover, z^α maps the upper half-plane into itself; actually it contracts all arguments by the factor α. Apart from some technical complications this allows to surround the singularities of f_α—which are all in $(-\infty, 0)$—by an integration contour and to use Cauchy's theorem for the half-plane to bring us back to the representation (3).

Coming back to the fractional powers λ^α, $0 < \alpha < 1$, we derive yet another representation formula. First note that $\lambda^\alpha = \int_0^\lambda \alpha s^{-(1-\alpha)}\, ds$ and that the integrand $s^{-(1-\alpha)}$ is a Stieltjes function which can be expressed as in (4). Fubini's theorem and the elementary equality

$$\int_0^\lambda \frac{1}{t+s}\, ds = \log\left(1 + \frac{\lambda}{t}\right)$$

yield

$$\lambda^\alpha = \frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \log\left(1 + \frac{\lambda}{t}\right) t^{\alpha-1}\, dt. \tag{5}$$

This representation will be called the Thorin representation of λ^α. Not every complete Bernstein function has a Thorin representation. The critical step in deriving (5) was the fact that the derivative of λ^α is a Stieltjes function.

What has been explained for fractional powers can be extended in various directions. On the level of functions, the structure of (1) is characteristic for the class $\mathcal{BF}$ of Bernstein functions, (3) for the class $\mathcal{CBF}$ of complete Bernstein functions, and (5) for the Thorin–Bernstein functions $\mathcal{TBF}$. If we consider $\exp(-tf)$ with f from $\mathcal{BF}$, $\mathcal{CBF}$ or $\mathcal{TBF}$, we are led to the corresponding families of completely monotone functions and measures. Apart from some minor conditions, these are the infinitely divisible distributions ID, the Bondesson class of measures BO and the generalized Gamma convolutions GGC. The diagrams in Remark 9.17 illustrate these connections. If we replace (formally) λ by $-A$, where A is a negative semi-definite matrix or a dissipative closed operator, then we get from (1) and (2) the classical formulae for fractional powers, while (3) turns into Balakrishnan's formula. Considering $\mathcal{BF}$ and $\mathcal{CBF}$ we obtain a fully-fledged functional calculus for generators and potential operators. Since complete Bernstein functions are operator monotone functions we can even recover the famous Heinz–Kato inequality.

Let us briefly describe the content and the structure of the book. It consists of three parts. The first part, Chapters 1–10, introduces the basic classes of functions: the positive definite functions comprising the completely monotone, Stieltjes and Hirsch functions, and the negative definite functions which consist of the Bernstein functions and their subfamilies—special, complete and Thorin–Bernstein functions. Two probabilistic intermezzi explore the connection between Bernstein functions and certain classes of probability measures. Roughly speaking, for every Bernstein function f the functions $\exp(-tf)$, $t > 0$, are completely monotone, which implies that $\exp(-tf)$ is the Laplace transform of an infinitely divisible sub-probability measure. This part of the book is essentially self-contained and should be accessible to non-specialists and graduate students.

In the second part of the book, Chapter 11 through Chapter 14, we turn to applications of Bernstein and complete Bernstein functions. The choice of topics reflects our own interests and is by no means complete. Notable omissions are applications in integral equations and continued fractions.

Among the topics are the spectral theorem for self-adjoint operators in a Hilbert space and a characterization of all functions which preserve the order (in quadratic form sense) of dissipative operators. Bochner's subordination plays a fundamental role in Chapter 12 where also a functional calculus for subordinate generators is developed. This calculus generalizes many formulae for fractional powers of closed operators. As another application of Bernstein and complete Bernstein functions we establish estimates for the eigenvalues of subordinate Markov processes. This is continued in Chapter 13 which contains a detailed study of excessive functions of killed and subordinate killed Brownian motion. Finally, Chapter 14 is devoted to two results in the theory of generalized diffusions, both related to complete Bernstein functions through Kreĭn's theory of strings. Many of these results appear for the first time in a monograph.

The third part of the book is formed by extensive tables of complete Bernstein functions. The main criteria for inclusion in the tables were the availability of explicit representations and the appearance in mathematical literature.

In the appendix we collect, for the readers' convenience, some supplementary results.

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Index of notation

This index is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below; numbers following an entry are page numbers.

Unless otherwise stated, binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limiting relations $f_j \xrightarrow{j\to\infty} f$, $\lim_j f_j$, $\liminf_j f_j$, $\limsup_j f_j$, $\sup_j f_j$ or $\inf_j f_j$ are always understood pointwise.

Operations and operators

$a \vee b$	maximum of a and b
$a \wedge b$	minimum of a and b
$\mathscr{L}$	Laplace transform, 1

Sets

$\mathbb{H}^{\uparrow}$	$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$
$\mathbb{H}^{\downarrow}$	$\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$
$\overrightarrow{\mathbb{H}}$	$\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$
$\mathbb{N}$	natural numbers: $1, 2, 3, \dots$
positive	always in the sense > 0
negative	always in the sense < 0

Spaces of functions

$\mathcal{B}$	Borel measurable functions
C	continuous functions
$\mathbb{H}$	harmonic functions, 179
$\mathbb{S}$	excessive functions, 178
$\mathcal{BF}$	Bernstein functions, 15
$\mathcal{CBF}$	complete Bernstein fns, 49
$\mathcal{CM}$	completely monotone fns, 2
$\mathcal{H}$	Hirsch functions, 105
$\mathcal{P}$	potentials, 45
$\mathcal{S}$	Stieltjes functions, 11
$\mathcal{SBF}$	special Bernstein fns, 92
$\mathcal{TBF}$	Thorin–Bernstein fns, 73

Sub- and superscripts

$+$	*sets:* non-negative elements, *functions:* non-negative part
$*$	non-trivial elements ($\not\equiv 0$)
$\perp$	orthogonal complement
b	bounded
c	compact support
f	subordinate w.r.t. the Bernstein function f

Spaces of distributions

BO	Bondesson class, 80
CE	convolutions of Exp, 87
Exp	exponential distributions, 88
GGC	generalized Gamma convolutions, 84
ID	infinitely divisible distr., 37
ME	mixtures of Exp, 81
SD	self-decomposable distr., 41

Contents

Chapter 1

Laplace transforms and completely monotone functions

In this chapter we collect some preliminary material which we need later on in order to study Bernstein functions.

As usual, we define the *(one-sided) Laplace transform* of a function $m : [0,\infty) \to [0,\infty)$ or a measure μ on the half-line $[0,\infty)$ by

$$\mathscr{L}(m;\lambda) := \int_0^\infty e^{-\lambda t} m(t)\,dt \quad \text{or} \quad \mathscr{L}(\mu;\lambda) := \int_{[0,\infty)} e^{-\lambda t}\,\mu(dt), \tag{1.1}$$

respectively, whenever these integrals converge. Obviously, $\mathscr{L}m = \mathscr{L}\mu_m$ if $\mu_m(dt)$ denotes the measure $m(t)\,dt$.

The following real-analysis lemma is helpful in order to show that finite measures are uniquely determined in terms of their Laplace transforms.

Lemma 1.1. *We have for all* $t, x \geqslant 0$

$$\lim_{\lambda\to\infty} e^{-\lambda t} \sum_{k\leqslant\lambda x} \frac{(\lambda t)^k}{k!} = \mathbb{1}_{[0,x]}(t). \tag{1.2}$$

Proof. Let us rewrite (1.2) in probabilistic terms: if X is a Poisson random variable with parameter λt, (1.2) states that

$$\lim_{\lambda\to\infty} \mathbb{P}(X \leqslant \lambda x) = \mathbb{1}_{[0,x]}(t).$$

From the basic formulae for the mean value and the variance of Poisson random variables, $\mathbb{E}X = \lambda t$ and $\operatorname{Var}X = \mathbb{E}((X-\lambda t)^2) = \lambda t$, we find for $t > x$ with Chebyshev's inequality

$$\begin{aligned}
\mathbb{P}(X \leqslant \lambda x) &\leqslant \mathbb{P}\big(|X-\lambda t| \geqslant \lambda(t-x)\big) \\
&\leqslant \frac{\mathbb{E}((X-\lambda t)^2)}{\lambda^2(t-x)^2} \\
&= \frac{\lambda t}{\lambda^2(t-x)^2} \xrightarrow{\lambda\to\infty} 0.
\end{aligned}$$

If $t \leqslant x$, a similar calculation yields

$$\begin{aligned}\mathbb{P}(X \leqslant \lambda x) &= 1 - \mathbb{P}\big(X - \lambda t > \lambda(x-t)\big)\\ &\geqslant 1 - \mathbb{P}\big(|X - \lambda t| > \lambda(x-t)\big) \xrightarrow{\lambda\to\infty} 1 - 0,\end{aligned}$$

and the claim follows. □

Proposition 1.2. *A measure μ supported in $[0,\infty)$ is finite if, and only if, $\mathscr{L}(\mu; 0+) < \infty$. The measure μ is uniquely determined by its Laplace transform.*

Proof. The first part of the assertion follows from monotone convergence since we have $\mu[0,\infty) = \int_{[0,\infty)} 1\, d\mu = \lim_{\lambda\to 0} \int_{[0,\infty)} e^{-\lambda t}\, \mu(dt)$.

For the uniqueness part we use first the differentiation lemma for parameter dependent integrals to get

$$(-1)^k \mathscr{L}^{(k)}(\mu;\lambda) = \int_{[0,\infty)} e^{-\lambda t}\, t^k\, \mu(dt).$$

Therefore,

$$\begin{aligned}\sum_{k\leqslant\lambda x} (-1)^k \mathscr{L}^{(k)}(\mu;\lambda)\frac{\lambda^k}{k!} &= \sum_{k\leqslant\lambda x}\int_{[0,\infty)} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\,\mu(dt)\\ &= \int_{[0,\infty)} \sum_{k\leqslant\lambda x} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\,\mu(dt)\end{aligned}$$

and we conclude with Lemma 1.1 and dominated convergence that

$$\lim_{\lambda\to\infty} \sum_{k\leqslant\lambda x} (-1)^k \mathscr{L}^{(k)}(\mu;\lambda)\frac{\lambda^k}{k!} = \int_{[0,\infty)} \mathbb{1}_{[0,x]}(t)\,\mu(dt) = \mu[0,x]. \qquad (1.3)$$

This shows that μ can be recovered from (all derivatives of) its Laplace transform. □

It is possible to characterize the range of Laplace transforms. For this we need the notion of complete monotonicity.

Definition 1.3. A function $f : (0,\infty) \to \mathbb{R}$ is a *completely monotone function* if f is of class C^∞ and

$$(-1)^n f^{(n)}(\lambda) \geqslant 0 \quad \text{for all } n \in \mathbb{N}\cup\{0\} \text{ and } \lambda > 0. \qquad (1.4)$$

The family of all completely monotone functions will be denoted by $\mathcal{CM}$.

The conditions (1.4) are often referred to as *Bernstein–Hausdorff–Widder conditions*. The next theorem is known as *Bernstein's theorem*.

The version given below appeared for the first time in [34] and independently in [287]. Subsequent proofs were given in [98] and [86]. The theorem may be also considered as an example of the general integral representation of points in a convex cone by means of its extremal elements. See Theorem 4.8 and [69] for an elementary exposition. The following short and elegant proof is taken from [212].

Theorem 1.4 (Bernstein). *Let $f : (0, \infty) \to \mathbb{R}$ be a completely monotone function. Then it is the Laplace transform of a unique measure μ on $[0, \infty)$, i.e. for all $\lambda > 0$,*

$$f(\lambda) = \mathscr{L}(\mu; \lambda) = \int_{[0,\infty)} e^{-\lambda t}\, \mu(dt).$$

Conversely, whenever $\mathscr{L}(\mu; \lambda) < \infty$ for every $\lambda > 0$, $\lambda \mapsto \mathscr{L}(\mu; \lambda)$ is a completely monotone function.

Proof. Assume first that $f(0+) = 1$ and $f(+\infty) = 0$. Let $\lambda > 0$. For any $a > 0$ and any $n \in \mathbb{N}$, we see by Taylor's formula

$$\begin{aligned} f(\lambda) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\lambda - a)^k + \int_a^\lambda \frac{f^{(n)}(s)}{(n-1)!} (\lambda - s)^{n-1}\, ds \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(a)}{k!} (a - \lambda)^k + \int_\lambda^a \frac{(-1)^n f^{(n)}(s)}{(n-1)!} (s - \lambda)^{n-1}\, ds. \end{aligned} \tag{1.5}$$

If $a > \lambda$, then by the assumption all terms are non-negative. Let $a \to \infty$. Then

$$\begin{aligned} \lim_{a\to\infty} \int_\lambda^a \frac{(-1)^n f^{(n)}(s)}{(n-1)!} (s - \lambda)^{n-1}\, ds &= \int_\lambda^\infty \frac{(-1)^n f^{(n)}(s)}{(n-1)!} (s - \lambda)^{n-1}\, ds \\ &\leqslant f(\lambda). \end{aligned}$$

This implies that the sum in (1.5) converges for every $n \in \mathbb{N}$ as $a \to \infty$. Thus, every term converges as $a \to \infty$ to a non-negative limit. For $n \geqslant 0$ let

$$\rho_n(\lambda) = \lim_{a\to\infty} \frac{(-1)^n f^{(n)}(a)}{n!} (a - \lambda)^n.$$

This limit does not depend on $\lambda > 0$. Indeed, for $\kappa > 0$,

$$\begin{aligned} \rho_n(\kappa) &= \lim_{a\to\infty} \frac{(-1)^n f^{(n)}(a)}{n!} (a - \kappa)^n \\ &= \lim_{a\to\infty} \frac{(-1)^n f^{(n)}(a)}{n!} (a - \lambda)^n \frac{(a - \kappa)^n}{(a - \lambda)^n} = \rho_n(\lambda). \end{aligned}$$

Let $c_n = \sum_{k=0}^{n-1} \rho_k(\lambda)$. Then

$$f(\lambda) = c_n + \int_\lambda^\infty \frac{(-1)^n f^{(n)}(s)}{(n-1)!} (s-\lambda)^{n-1}\, ds.$$

Clearly, $f(\lambda) \geqslant c_n$ for all $\lambda > 0$. Let $\lambda \to \infty$. Since $f(+\infty) = 0$, it follows that $c_n = 0$ for every $n \in \mathbb{N}$. Thus we have obtained the following integral representation of the function f:

$$f(\lambda) = \int_\lambda^\infty \frac{(-1)^n f^{(n)}(s)}{(n-1)!} (s-\lambda)^{n-1}\, ds. \tag{1.6}$$

By the monotone convergence theorem

$$1 = \lim_{\lambda\to 0} f(\lambda) = \int_0^\infty \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}\, ds. \tag{1.7}$$

Let

$$f_n(s) = \frac{(-1)^n}{n!} f^{(n)}\left(\frac{n}{s}\right)\left(\frac{n}{s}\right)^{n+1}. \tag{1.8}$$

Using (1.7) and changing variables according to s/t, it follows that for every $n \in \mathbb{N}$, f_n is a probability density function on $(0,\infty)$. Moreover, the representation (1.6) can be rewritten as

$$\begin{aligned} f(\lambda) &= \int_0^\infty \left(1-\frac{\lambda}{s}\right)_+^{n-1} \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}\, ds \\ &= \int_0^\infty \left(1-\frac{\lambda t}{n}\right)_+^{n-1} f_n(t)\, dt. \end{aligned} \tag{1.9}$$

By Helly's selection theorem, Corollary A.8, there exist a subsequence $(n_k)_{k\geqslant 1}$ and a probability measure μ on $(0,\infty)$ such that $f_{n_k}(t)\, dt$ converges weakly to $\mu(dt)$. Further, for every $\lambda > 0$,

$$\lim_{n\to\infty} \left(1-\frac{\lambda t}{n}\right)_+^{n-1} = e^{-\lambda t}$$

uniformly in $t \in (0,\infty)$. By taking the limit in (1.9) along the subsequence $(n_k)_{k\geqslant 1}$, it follows that

$$f(\lambda) = \int_{(0,\infty)} e^{-\lambda t}\, \mu(dt).$$

Uniqueness of μ follows from Proposition 1.2.

Assume now that $f(0+) < \infty$ and $f(+\infty) = 0$. By looking at $f/f(0+)$ we see that the representing measure for f is uniquely given by $f(0+)\mu$.

Now let f be an arbitrary completely monotone function with $f(+\infty) = 0$. For every $a > 0$, define $f_a(\lambda) := f(\lambda + a)$, $\lambda > 0$. Then f_a is a completely monotone function with $f_a(0+) = f(a) < \infty$ and $f_a(+\infty) = 0$. By what has been already proved, there exists a unique finite measure μ_a on $(0, \infty)$ such that $f_a(\lambda) = \int_{(0,\infty)} e^{-\lambda t}\, \mu_a(dt)$. It follows easily that for $b > 0$ we have $e^{at}\mu_a(dt) = e^{bt}\mu_b(dt)$. This shows that we can consistently define the measure μ on $(0, \infty)$ by $\mu(dt) = e^{at}\mu_a(dt)$, $a > 0$. In particular, the representing measure μ is uniquely determined by f. Now, for $\lambda > 0$,

$$f(\lambda) = f_{\lambda/2}(\lambda/2) = \int_{(0,\infty)} e^{(-\lambda/2)t}\, \mu_{\lambda/2}(dt)$$
$$= \int_{(0,\infty)} e^{-\lambda t} e^{(\lambda/2)t}\, \mu_{\lambda/2}(dt) = \int_{(0,\infty)} e^{-\lambda t}\, \mu(dt).$$

Finally, if $f(+\infty) = c > 0$, add $c\delta_0$ to μ.

For the converse we set $f(\lambda) := \mathscr{L}(\mu;\lambda)$. Fix $\lambda > 0$ and pick $\epsilon \in (0, \lambda)$. Since $t^n = \epsilon^{-n}(\epsilon t)^n \leqslant n!\epsilon^{-n}e^{\epsilon t}$ for all $t > 0$, we find

$$\int_{[0,\infty)} t^n\, e^{-\lambda t}\, \mu(dt) \leqslant \frac{n!}{\epsilon^n} \int_{[0,\infty)} e^{-(\lambda-\epsilon)t}\, \mu(dt) = \frac{n!}{\epsilon^n}\, \mathscr{L}(\mu;\lambda-\epsilon)$$

and this shows that we may use the differentiation lemma for parameter dependent integrals to get

$$(-1)^n f^{(n)}(\lambda) = (-1)^n \int_{[0,\infty)} \frac{d^n}{d\lambda^n} e^{-\lambda t}\, \mu(dt) = \int_{[0,\infty)} t^n\, e^{-\lambda t}\, \mu(dt) \geqslant 0. \quad \square$$

Remark 1.5. The last formula in the proof of Theorem 1.4 shows, in particular, that $f^{(n)}(\lambda) \neq 0$ for all $n \geqslant 1$ and all $\lambda > 0$ unless $f \in \mathcal{CM}$ is identically constant.

Corollary 1.6. *The set $\mathcal{CM}$ of completely monotone functions is a convex cone, i.e.*

$$sf_1 + tf_2 \in \mathcal{CM} \quad \textit{for all} \quad s,t \geqslant 0 \quad \textit{and} \quad f_1, f_2 \in \mathcal{CM},$$

which is closed under multiplication, i.e.

$$\lambda \mapsto f_1(\lambda)f_2(\lambda) \quad \textit{is in} \quad \mathcal{CM} \quad \textit{for all} \quad f_1, f_2 \in \mathcal{CM},$$

and under pointwise convergence:

$$\mathcal{CM} = \overline{\{\mathscr{L}\mu \,:\, \mu \textit{ is a finite measure on } [0,\infty)\}}$$

(the closure is taken with respect to pointwise convergence).

Proof. That $\mathcal{CM}$ is a convex cone follows immediately from the definition of a completely monotone function or, alternatively, from the representation formula in Theorem 1.4.

If μ_j denotes the representing measure of f_j, $j = 1, 2$, the convolution

$$\mu[0,u] := \mu_1 \star \mu_2[0,u] := \iint_{[0,\infty)\times[0,\infty)} \mathbb{1}_{[0,u]}(s+t)\,\mu_1(ds)\mu_2(dt)$$

is the representing measure of the product $f_1 f_2$. Indeed,

$$\int_{[0,\infty)} e^{-\lambda u}\,\mu(du) = \int_{[0,\infty)}\int_{[0,\infty)} e^{-\lambda(s+t)}\,\mu_1(ds)\mu_2(dt) = f_1(\lambda)f_2(\lambda).$$

Write $M := \{\mathscr{L}\mu \;:\; \mu \text{ is a finite measure on } [0,\infty)\}$. Theorem 1.4 shows that $M \subset \mathcal{CM} \subset \overline{M}$. We are done if we can show that $\mathcal{CM}$ is closed under pointwise convergence. For this choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{CM}$ such that $\lim_{n\to\infty} f_n(\lambda) = f(\lambda)$ exists for every $\lambda > 0$. If μ_n denotes the representing measure of f_n, we find for every $a > 0$

$$\mu_n[0,a] \leqslant e^{a\lambda}\int_{[0,a]} e^{-\lambda t}\,\mu_n(dt) \leqslant e^{a\lambda} f_n(\lambda) \xrightarrow{n\to\infty} e^{a\lambda} f(\lambda)$$

which means that the family of measures $(\mu_n)_{n\in\mathbb{N}}$ is bounded in the vague topology, hence vaguely sequentially compact, see Appendix A.1. Thus, there exist a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and some measure μ such that $\mu_{n_k} \to \mu$ vaguely. For $\chi \in C_c[0,\infty)$ with $0 \leqslant \chi \leqslant 1$, we find

$$\int_{[0,\infty)} \chi(t)e^{-\lambda t}\,\mu(dt) = \lim_{k\to\infty}\int_{[0,\infty)} \chi(t)e^{-\lambda t}\,\mu_{n_k}(dt) \leqslant \liminf_{k\to\infty} f_{n_k}(\lambda) = f(\lambda).$$

Taking the supremum over all such χ, we can use monotone convergence to get

$$\int_{[0,\infty)} e^{-\lambda s}\,\mu(dt) \leqslant f(\lambda).$$

On the other hand, we find for each $a > 0$

$$\begin{aligned} f_{n_k}(\lambda) &= \int_{[0,a)} e^{-\lambda t}\,\mu_{n_k}(dt) + \int_{[a,\infty)} e^{-\frac{1}{2}\lambda t}e^{-\frac{1}{2}\lambda t}\,\mu_{n_k}(dt) \\ &\leqslant \int_{[0,a)} e^{-\lambda t}\,\mu_{n_k}(dt) + e^{-\frac{1}{2}\lambda a} f_{n_k}\left(\frac{1}{2}\lambda\right). \end{aligned}$$

If we let $k \to \infty$ and then $a \to \infty$ along a sequence of continuity points of μ we get $f(\lambda) \leqslant \int_{[0,\infty)} e^{-\lambda t}\,\mu(dt)$ which shows that $f \in \mathcal{CM}$ and that the measure μ is actually independent of the particular subsequence. In particular, $\mu = \lim_{n\to\infty}\mu_n$ vaguely in the space of measures supported in $[0,\infty)$. □