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# GLOBAL LORENTZIAN GEOMETRY

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# PREFACE

This book is about Lorentzian geometry, the mathematical theory used in general relativity, treated from the viewpoint of global differential geometry. Our goal is to help bridge the gap between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of global Lorentzian geometry. The growing importance in physics of this approach is clearly illustrated by the recent Hawking-Penrose singularity theorems described in the text of Hawking and Ellis (1973).

The Lorentzian distance function is used as a unifying concept in our book. Furthermore, we frequently compare and contrast the results and techniques of Lorentzian geometry to those of Riemannian geometry to alert the reader to the basic differences between these two geometries.

This book has been written especially for the mathematician who has a basic acquaintance with Riemannian geometry and wishes to learn Lorentzian geometry. Accordingly, this book is written using the notation and methods of modern differential geometry. For readers less familiar with this notation, we have included Appendix A which gives the local coordinate representations for the symbols used.

The basic prerequisites for this book are a working knowledge of general topology and differential geometry. Thus this book should be accessible to advanced graduate students in either mathematics or mathematical physics.

In writing this monograph, both authors profited greatly from the opportunity to lecture on part of this material during the spring semester, 1978, at the University of Missouri-Columbia. The second author also gave a series of lectures on this material in Ernst Ruh's seminar in differential geometry at Bonn University during the summer semester, 1978, and would like to thank Professor Ruh for giving him the opportunity to speak on this material. We would like to thank C. Ahlbrandt, D. Carlson, and M. Jacobs for several helpful conversations on Section 2.4 and the calculus of variations. We would like to thank M. Engman, S. Harris, K. Nomizu, T. Powell, D. Retzloff, and H. Wu for helpful comments on our preliminary version of this monograph. We also thank S. Harris for contributing Appendix D to this monograph and J.-H. Eschenburg for calling our attention to the Diplomarbeit of Bölts (1977). To anyone who has read either of the excellent books of Gromoll, Klingenberg, and Meyer (1975) on Riemannian manifolds or of Hawking and Ellis (1973) on general relativity, our debt to these authors in writing this work will be obvious. It is also a pleasure for both authors to thank the Research Council of the University of Missouri-Columbia and for the second author to thank the Sonderforschungsbereich Theoretische Mathematik 40 of the Mathematics Department, Bonn University, and to acknowledge an NSF Grant MCS77-18723(02) held at the Institute for Advanced Study, Princeton, New Jersey, for partial financial support while we were working on this monograph. Finally it is a pleasure to thank Diane Coffman, DeAnna Williamson, and Debra Retzloff for the patient and cheerful typing of the manuscript.

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CONTENTS

Chapter 1

# INTRODUCTION: RIEMANNIAN THEMES IN LORENTZIAN GEOMETRY

Recent progress on causality theory, singularity theory, and black holes in general relativity described in the influential text of Hawking and Ellis (1973) has resulted in a resurgence of interest in global Lorentzian geometry. Indeed, a better understanding of global Lorentzian geometry was required for the development of singularity theory. For example, it was necessary to know that causally related points in globally hyperbolic subsets of space-times could be joined by a nonspacelike geodesic segment maximizing the Lorentzian arc length among all nonspacelike curves joining the two given points. In addition, much work done in the 1970s on foliating asymptotically flat Lorentzian manifolds by families of maximal hypersurfaces has been motivated by general relativity [cf. Choquet-Bruhat, Fisher, and Marsden (1979) for a partial list of references].

All of these results naturally suggest that a systematic study of global Lorentzian geometry should be made. The development of "modern" global Riemannian geometry as described in any of the standard texts [cf. Bishop and Crittenden (1964), Gromoll, Klingenberg, and Meyer (1975), Helgason (1962), Hicks (1965)] supports the idea that a comprehensive treatment of global Lorentzian geometry should be grounded in three fundamental topics: geodesic and metric completeness, the Lorentzian distance function, and a Morse index theory valid for nonspacelike geodesic segments in an arbitrary Lorentzian manifold.

Geodesic completeness, or more accurately, geodesic incompleteness, has played a crucial role in the development of singularity theory in general relativity and has been thoroughly explored within this framework. However the Lorentzian distance function has not been as well investigated, although it has been of some use in the study of singularities [cf. Hawking (1967), Hawking and Ellis (1973), Tipler (1977a), Beem and Ehrlich (1979a)]. Some of the properties of the Lorentzian distance function needed in general relativity are briefly described in Hawking and Ellis (1973, pp. 215-217). Further results relating Lorentzian distance to causality and the global behavior of nonspacelike geodesics have been given in Beem and Ehrlich (1979b).

Uhlenbeck (1975), Everson and Talbot (1976), and Woodhouse (1976) have studied Morse index theory for globally hyperbolic space-times and we have sketched [cf. Beem and Ehrlich (1979 c,d)] a Morse index theory for nonspacelike geodesics in arbitrary space-times. But no complete treatment of this theory for arbitrary space-times has been published previously.

It is the purpose of this monograph to first review known results on geodesic and metric completeness. Then we give a detailed treatment of the Lorentzian distance function and of the Morse index theory for nonspacelike geodesics in arbitrary space-times. Finally we show how these concepts may be applied to global Lorentzian geometry and singularity theory in general relativity.

The Lorentzian distance function has many similarities with the Riemannian distance function but also many differences. Since the Lorentzian distance function is not so well known, we now review the main properties of the Riemannian distance function, then compare and contrast the corresponding results for the Lorentzian distance function.

For the rest of this portion of the introduction, we will let $(N,g_0)$ denote a Riemannian manifold and $(M,g)$ denote a Lorentzian manifold, respectively.

Thus N is a smooth paracompact manifold equipped with a positive definite inner product $g_0|_p : T_pN \times T_pN \longrightarrow \mathbb{R}$ on each tangent space $T_pN$. In addition, if X and Y are arbitrary smooth vector fields on N, the function $N \longrightarrow \mathbb{R}$ given by $p \longrightarrow g_0(X(p),Y(p))$ is required to be a smooth function. The Riemannian structure $g_0 : TN \times TN \longrightarrow \mathbb{R}$ then defines the Riemannian distance function

$$d_0 : N \times N \longrightarrow [0,\infty)$$

as follows. Let $\Omega_{p,q}$ denote the set of piecewise smooth curves in N from p to q. Given $c \in \Omega_{p,q}$, $c : [0,1] \longrightarrow N$, there is a finite partition $0 = t_1 < t_2 < \cdots < t_k = 1$ such that $c \mid [t_i,t_{i+1}]$ is smooth for each i. The Riemannian arc length of c with respect to $g_0$ is defined as

$$L_0(c) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g_0(c'(t),c'(t))}\, dt$$

The Riemannian distance $d_0(p,q)$ between p and q is then defined to be

$$d_0(p,q) = \inf\{L_0(c) : c \in \Omega_{p,q}\} \geq 0$$

For any Riemannian metric $g_0$ for N, the function $d_0 : N \times N \longrightarrow [0,\infty)$ has the following properties:

(1) $d_0(p,q) = d_0(q,p)$ for all $p,q \in N$.

(2) $d_0(p,q) \leq d_0(p,r) + d_0(r,q)$ for all $p,q,r \in N$.

(3) $d_0(p,q) = 0$ iff $p = q$.

More surprisingly,

(4) $d_0 : N \times N \longrightarrow [0,\infty)$ is continuous and the family of metric balls

$$B(p,\varepsilon) = \{q \in N : d(p,q) < \varepsilon\}$$

for all $p \in N$ and $\varepsilon > 0$ forms a basis for the given manifold topology.

Thus the metric topology and the given manifold topology coincide. Furthermore, by a result of Whitehead (1932), given any $p \in N$, there exists an $R > 0$ such that for any $\varepsilon$ with $0 < \varepsilon < R$, the metric ball $B(p,\varepsilon)$ is geodesically convex. Thus for any $\varepsilon$ with $0 < \varepsilon < R$, the set $B(p,\varepsilon)$ is diffeomorphic to the n disk, $n = \dim(N)$, and the set $\{q \in N : d(p,q) = \varepsilon\}$ is diffeomorphic to $S^{n-1}$.

Removing the origin from $\mathbb{R}^2$ equipped with the usual Euclidean metric and setting $p = (-1,0)$, $q = (1,0)$, one calculates that $d_0(p,q) = 2$, but finds no curve $c \in \Omega_{p,q}$ with $L_0(c) = d_0(p,q)$ and also no smooth geodesic from p to q.

Thus the following questions arise naturally. Given a manifold N, find conditions on a Riemannian metric $g_0$ for N such that

(i) All geodesics in N may be extended to be defined on all of $\mathbb{R}$.

(ii) The pair $(N,d_0)$ is a complete metric space in the sense that all Cauchy sequences converge.

(iii) Given any two points $p,q \in N$, there is a smooth geodesic segment $c \in \Omega_{p,q}$ with $L_0(c) = d_0(p,q)$.

A distance realizing geodesic segment as in (iii) is called a *minimal* geodesic segment. The word *minimal* is used here since the definition of Riemannian distance implies that $L_0(\gamma) \geqslant d_0(p,q)$ for all $\gamma \in \Omega_{p,q}$. More generally, one may define an arbitrary piecewise smooth curve $\gamma \in \Omega_{p,q}$ to be *minimal* if $L_0(\gamma) = d_0(p,q)$. Using the variation theory of the arc length functional, it may be shown that if $\gamma \in \Omega_{p,q}$ is minimal, then $\gamma$ may be reparameterized to a smooth geodesic segment.

The question of finding criteria on $g_0$ such that (i), (ii), or (iii) hold was resolved by H. Hopf and W. Rinow in their famous paper (1931). In modern terminology the Hopf-Rinow theorem asserts the following:

HOPF-RINOW THEOREM For any Riemannian manifold $(N,g_0)$ the following are equivalent:

(a) Metric completeness: $(N,d_0)$ is a complete metric space.

(b) Geodesic completeness: For any $v \in TN$, the geodesic $c(t)$ in N with $c'(0) = v$ is defined for all positive and negative real numbers $t \in \mathbb{R}$.

(c) For some $p \in N$, the exponential map $\exp_p$ is defined on the entire tangent space $T_pN$ to N at p.

(d) Finite compactness: Every subset K of N that is $d_0$ bounded (i.e., $\sup\{d_0(p,q) : p,q \in K\} < \infty$) has compact closure.

Furthermore, if any of (a) through (d) holds, then

(e) Given any $p,q \in N$, there exists a smooth geodesic segment c from p to q with $L_0(c) = d_0(p,q)$.

A Riemannian manifold $(N,g_0)$ is said to be *complete* provided any one (and hence all) of conditions (a) through (d) is satisfied. It should be stressed that the Hopf-Rinow theorem guarantees the equivalence of metric and geodesic completeness and also that *all* Riemannian metrics for a compact smooth manifold are complete. Unfortunately, *none* of these statements are valid for arbitrary Lorentzian manifolds.

A remaining question for noncompact but paracompact manifolds is the existence of complete Riemannian metrics. This was settled by Nomizu and Ozeki's (1961) proof that given any Riemannian metric $g_0$ for N, there is a *complete* Riemannian metric for N globally conformal to $g_0$. Since any paracompact, connected smooth manifold N admits a Riemannian metric by a partition of unity argument, N also admits a complete Riemannian metric.

We now turn our attention to the Lorentzian manifold (M,g). A Lorentzian metric g for the smooth paracompact manifold M is the assignment of a nondegenerate bilinear form $g|_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ with diagonal form (-, +, ..., +) to each tangent space. It is well known that if M is compact and $\chi(M) \neq 0$, then M admits *no* Lorentzian metrics. On the other hand, any noncompact manifold admits a Lorentzian metric. Geroch (1968a) and Marante (1972) have also shown that a smooth Hausdorff manifold which admits a Lorentzian metric is paracompact.

Nonzero tangent vectors are classified as *timelike*, *spacelike*, *nonspacelike*, or *null*, respectively, according to whether $g(v,v) < 0$, resp., $> 0$, $\leq 0$, $= 0$. [Some authors use the convention (+, -, ..., -) for the Lorentzian metric and hence all of the inequality signs in the above definition are reversed for them.] A Lorentzian manifold

(M,g) is said to be *time oriented* if M admits a continuous, nowhere vanishing timelike vector field X. This vector field is used to separate the nonspacelike vectors at each point into two classes, called *future directed* and *past directed*. A *space-time* is then a Lorentzian manifold (M,g) together with a choice of time orientation. We will usually work with space-times below.

In order to define the Lorentzian distance function and discuss its properties, we need to introduce some concepts from elementary causality theory. It is standard to write $p \ll q$ if there is a future-directed piecewise smooth timelike curve in M from p to q, and $p \leqslant q$ if $p = q$ or if there is a future directed piecewise smooth nonspacelike curve in M from p to q. The *chronological past* and *future* of p are then given respectively by $I^-(p) = \{q \in M : q \ll p\}$ and $I^+(p) = \{q \in M : p \ll q\}$. The *causal past* and *future* of p are defined as $J^-(p) = \{q \in M : q \leqslant p\}$ and $J^+(p) = \{q \in M : p \leqslant q\}$. The sets $I^-(p)$ and $I^+(p)$ are always open in any space-time, but the sets $J^-(p)$ and $J^+(p)$ are neither open nor closed in general (cf. Figure 1.1).

The *causal structure* of the space-time (M,g) may be defined as the collection of past and future sets at all points of M together with their properties. It may be shown that two strongly causal Lorentzian metrics $g_1$ and $g_2$ for M determine the same past and future sets at all points iff the two metrics are globally conformal [i.e., $g_1 = \Omega g_2$ for some smooth function $\Omega : M \longrightarrow (0,\infty)$]. Letting C(M,g) denote the set of Lorentzian metrics globally conformal to g, it follows that properties suitably defined using the past and future sets hold simultaneously either for all metrics in C(M,g) or for no metrics in C(M,g). Thus all of the basic properties of elementary causality theory depend only on the conformal class C(M,g) and not on the choice of Lorentzian metric representing C(M,g).

Perhaps the two most elementary properties to require of the conformal structure C(M,g) are either that (M,g) be chronological or that (M,g) be causal. A space-time (M,g) is said to be