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HANDBOOK OF STATISTICAL DISTRIBUTIONS

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Handbook of Statistical Distributions

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Preface

This handbook is the outcome of an attempt to collect, in as concise a form as is possible, many results which statisticians find useful. In so doing the authors have not always quoted original sources of the discoveries given, but rather the most readily available sources.

This text is not intended as a book of statistical theory but instead as a source book where concise statements and references may be found.

Throughout the text, all expressions given are regarded as being implicitly restricted to the case where all quantities contained therein exist and are finite. In cases where questions might arise, the authors have noted these restrictions.

It is hoped that students in graduate statistics courses will find this summary particularly useful. Research workers should also find the book to be helpful in tracing references to material they desire.

In the preparation of the manuscript the authors have checked and rechecked the information given. However, considering the diversity of the material, one hundred percent accuracy cannot be

achieved. Readers are cordially invited to contact the authors regarding any errors or suggestions for improvements in presentation.

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August, 1976

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CONTENTS

Preface	xiii
1. MOMENTS, CUMULANTS, AND GENERATING FUNCTIONS	
1.1 Moments and Cumulants	1
1.1.1 Definitions	1
1.1.2 Comments	4
1.1.3 Recurrence relations between moments	7
1.1.4 Useful formulas	8
1.1.5 The problem of moments	9
1.1.6 Some useful results	11
1.2 Generating Functions	12
1.2.1 Definitions	13
1.2.2 Comments	14
1.2.3 Relations between generating functions and moments	14
1.2.4 Useful formulas	15
1.3 More on Characteristic Functions	16
1.3.1 Conditions for a function to be a characteris- tic function	16
1.3.2 Comments	17
1.3.3 Some useful results	18

1.4	Moments, Cumulants, and Generating Functions for Some Useful Discrete Distributions	20
1.4.1	Binomial	20
1.4.2	Poisson	21
1.4.3	Negative binomial	22
1.4.4	Hypergeometric	23
1.4.5	Logarithmic series	25
1.4.6	Uniform	26
1.5	Moments, Cumulants, and Generating Functions for Some Useful Continuous Distributions	27
1.5.1	Normal	27
1.5.2	Lognormal	28
1.5.3	Inverse Gaussian	29
1.5.4	Cauchy	30
1.5.5	Gamma	30
1.5.6	Weibull	31
1.5.7	Beta	32
1.5.8	Power function	33
1.5.9	Uniform	33
1.5.10	Pareto	34
1.5.11	Extreme-value	35
1.5.12	Laplace	36
1.5.13	Logistic	37
1.5.14	Chi	38
1.5.15	Chi square	39
1.5.16	Student's t	40
1.5.17	F	41
1.5.18	Noncentral chi square	42
1.5.19	Noncentral t	43
1.5.20	Noncentral F	44
2.	INEQUALITIES	
2.1	Moment Inequalities	45
2.1.1	Some well-known inequalities	45
2.1.2	Some other inequalities	47
2.2	Chebyshev Inequalities	50
2.2.1	Inequalities using a single random variable	50
2.2.2	Inequalities using sums of random variables	55
3.	ORDER STATISTICS	
3.1	Order Statistics	61
3.1.1	Definition	61

3.1.2	Comments	62
3.2	Distribution of Order Statistics	62
3.2.1	Continuous distribution case	63
3.2.2	Recurrence relations between continuous distribution functions	64
3.2.3	Comments	65
3.2.4	Recurrence relations between expected values (continuous distribution case)	66
3.2.5	Comments	69
3.2.6	Discrete distribution case	70
3.2.7	Some useful formulas	71
3.2.8	Finite population case	72
3.2.9	Bounds for certain moments	73
3.3	Some Useful Results	74
3.4	Moments of Order Statistics from Some Discrete Distributions	75
3.4.1	Geometric	75
3.4.2	Negative binomial	76
3.5	Moments of Order Statistics from Some Continuous Distributions	76
3.5.1	Normal	76
3.5.2	Cauchy	77
3.5.3	Exponential	77
3.5.4	Gamma	78
3.5.5	Weibull	79
3.5.6	Power function	79
3.5.7	Uniform	80
3.5.8	Pareto	80
3.5.9	Extreme-value	81
3.5.10	Laplace	81
3.5.11	Logistic	82
3.5.12	Chi (1 df)	83
3.5.13	Triangular	84
4.	FAMILIES OF DISTRIBUTIONS	
4.1	Pearson (Continuous) Distributions	85
4.1.1	Definition	85
4.1.2	Properties	86
4.2	Exponential Family of Distributions	87
4.2.1	Definitions	87
4.2.2	Properties and some useful results	89

4.3	Linear Exponential Type (Continuous) Distributions (A subclass of the exponential family)	90
4.3.1	Definition	90
4.3.2	Properties	90
4.4	Pólya Type (Continuous) Distributions.	91
4.4.1	Definition	91
4.4.2	Properties	92
4.5	Monotone Likelihood Ratio Distributions	92
4.5.1	Definition	92
4.5.2	Properties (continuous distributions).	92
4.5.3	Some useful results (continuous distributions)	93
4.6	Generalized Power Series (Discrete) Distributions (GPSD)	94
4.6.1	Definition	94
4.6.2	Properties	94
4.6.3	Some useful results	95
4.7	Monotone Failure Rate (Continuous) Distributions	96
4.7.1	Definition	96
4.7.2	Properties	97
4.7.3	Some useful results	98
4.8	New Better (Worse) Than Used (Continuous) Distributions	99
4.8.1	Definitions	99
4.8.2	Properties and some useful results	99
4.9	Stable (Continuous) Distributions.	100
4.9.1	Definition	100
4.9.2	Properties	100
4.9.3	Some useful results	101
4.10	Infinitely Divisible Distributions (IDD).	102
4.10.1	Definition	102
4.10.2	Properties	103
4.10.3	Some useful results	103
4.11	Unimodal Distributions	104
4.11.1	Definition	104
4.11.2	Properties	104

4.11.3	Some useful results	105
4.12	Classification of Some Useful Discrete Distributions . .	105
4.12.1	Binomial	106
4.12.2	Poisson	106
4.12.3	Negative binomial	106
4.12.4	Hypergeometric	107
4.12.5	Logarithmic series	107
4.12.6	Uniform	107
4.13	Classification of Some Continuous Distributions	108
4.13.1	Normal	108
4.13.2	Lognormal	109
4.13.3	Inverse Gaussian	109
4.13.4	Cauchy	110
4.13.5	Gamma	110
4.13.6	Weibull	111
4.13.7	Beta	111
4.13.8	Power function	112
4.13.9	Uniform	113
4.13.10	Pareto	113
4.13.11	Extreme-value	113
4.13.12	Truncated extreme-value	114
4.13.13	Laplace	114
4.13.14	Logistic	115
4.13.15	Chi	115
5.	CHARACTERIZATION OF DISTRIBUTIONS	
5.1	General Characterizations	117
5.2	Characterization of Some Useful Discrete Distributions . .	122
5.2.1	Binomial	122
5.2.2	Poisson	122
5.2.3	Geometric	124
5.2.4	Negative binomial	125
5.2.5	Logarithmic series	125
5.3	Characterization of Some Useful Continuous Distributions	126
5.3.1	Normal	126
5.3.2	Inverse Gaussian	129
5.3.3	Cauchy	130
5.3.4	Exponential	131
5.3.5	Gamma	134
5.3.6	Weibull	136
5.3.7	Beta	136
5.3.8	Power function	137

5.3.9	Pareto	138
5.3.10	Extreme-value	139
5.3.11	Laplace	139
6.	POINT ESTIMATION	
6.1	Introduction	141
6.1.1	Some definitions	141
6.1.2	Unbiasedness	141
6.1.3	Comments	142
6.1.4	Consistency	143
6.1.5	Comments	143
6.1.6	Efficiency	144
6.2	Methods of Finding Estimators	144
6.2.1	Maximum-likelihood estimation	144
6.2.2	Properties	145
6.2.3	Method-of-moments	147
6.2.4	Comments	147
6.3	Sufficient Statistics	147
6.3.1	Equivalent definitions of sufficiency	148
6.3.2	Comments	148
6.3.3	Factorization criterion	149
6.3.4	Some useful results	149
6.4	Complete Families and Complete Statistics	151
6.4.1	Definitions	151
6.4.2	Comments	151
6.4.3	Some useful results	152
6.5	Uniformly Minimum-Variance Unbiased Estimator (UMVUE). . .	153
6.5.1	Definition	153
6.5.2	Comments	153
6.5.3	Some useful results	154
6.6	Complete Sufficient Statistics and UMVUE'S (Discrete). . .	157
6.6.1	Binomial	157
6.6.2	Poisson	158
6.6.3	Negative binomial	159
6.6.4	Hypergeometric	159
6.6.5	Logarithmic series	160
6.6.6	Uniform	160
6.6.7	Generalized power series.	161

6.7 Complete Sufficient Statistics and UMVUE'S (Continuous)	161
6.7.1 Normal	162
6.7.2 Lognormal	164
6.7.3 Inverse Gaussian	165
6.7.4 Cauchy	166
6.7.5 Truncated exponential	166
6.7.6 Gamma	168
6.7.7 Weibull	168
6.7.8 Beta	169
6.7.9 Power function	169
6.7.10 Uniform	170
6.7.11 Pareto	172
6.7.12 Truncated extreme-value	173
6.7.13 Chi	173
6.7.14 Burr	174
7. CONFIDENCE INTERVALS	
7.1 Introduction	175
7.1.1 Definitions	175
7.1.2 Comments	176
7.1.3 Criteria for selecting confidence intervals	177
7.1.4 Methods of finding confidence intervals	178
7.1.5 Some useful results	179
7.2 Confidence Intervals for Some Useful Discrete Distributions	180
7.2.1 Binomial	180
7.2.2 Poisson	181
7.2.3 Negative binomial	182
7.2.4 Multinomial	182
7.3 Confidence Intervals for Some Continuous Distributions	183
7.3.1 Normal	183
7.3.2 Lognormal	186
7.3.3 Inverse Gaussian	187
7.3.4 Cauchy	187
7.3.5 Exponential	188
7.3.6 Truncated exponential	189
7.3.7 Gamma	189
7.3.8 Weibull	190
7.3.9 Power function	191
7.3.10 Uniform	191
7.3.11 Pareto	192
7.3.12 Extreme-value	193

7.3.13	Truncated extreme-value	193
7.3.14	Laplace	193
7.3.15	Logistic	194
7.3.16	Chi	195
7.3.17	Burr	195
8. PROPERTIES OF DISTRIBUTIONS		
8.1	Discrete Distributions	197
8.1.1	Binomial	197
8.1.2	Poisson	201
8.1.3	Negative binomial	204
8.1.4	Hypergeometric	205
8.1.5	Logarithmic series	207
8.2	Continuous Distributions	208
8.2.1	Normal	208
8.2.2	Lognormal	210
8.2.3	Inverse Gaussian	211
8.2.4	Cauchy	212
8.2.5	Exponential	212
8.2.6	Gamma	214
8.2.7	Weibull	215
8.2.8	Beta	216
8.2.9	Uniform	218
8.2.10	Pareto	218
8.2.11	Extreme-value	219
8.2.12	Laplace	219
8.2.13	Logistic	220
8.2.14	Chi square	220
8.2.15	t	222
8.2.16	F	224
8.2.17	Noncentral chi square	225
8.2.18	Noncentral t	228
8.2.19	Noncentral F	229
8.2.20	Noncentral beta	232
9. BASIC LIMIT THEOREMS		
9.1	Types of Convergence	233
9.1.1	Definitions	233
9.1.2	Comments	234
9.1.3	Some other results	236
9.2	Laws of Large Numbers	236
9.2.1	Weak law of large numbers	237
9.2.2	Strong law of large numbers	238

9.3	The Central Limit Theorem	238
9.3.1	Independent random variables case	238
9.3.2	Dependent random variables case	239
9.3.3	Some other results	240
9.4	Some Other Limit Theorems	240
10.	MISCELLANEOUS RESULTS	
10.1	Gamma Functions	243
10.1.1	Definition	243
10.1.2	Properties	243
10.2	Incomplete Gamma Functions	244
10.2.1	Definition	244
10.2.2	Properties	244
10.3	Beta Functions	245
10.3.1	Definition	245
10.3.2	Properties	245
10.4	Incomplete Beta Functions	246
10.4.1	Definition	246
10.4.2	Properties	246
10.5	Convex (Concave) Functions	247
10.5.1	Definitions	247
10.5.2	Properties	247
10.6	Transformation of Statistics	248
10.6.1	Square root transformation of a Poisson random variable	248
10.6.2	Sin^{-1} transformation of the square root of a binomial random variable	249
10.6.3	Tanh^{-1} transformation of the correlation coefficient	249
10.6.4	Additional transformations	249
10.7	Stirling Numbers	250
10.7.1	Definitions	250
10.7.2	Properties	250

10.8	Bernoulli Numbers	251
10.9	Hypergeometric Functions	251
10.9.1	Definitions	251
10.9.2	Properties	251
10.10	The Notations O and o	252
10.11	K-Statistics	253
10.12	Some Useful Combinatorials	253
10.13	Some Useful Series	256
10.13.1	Taylor's series	256
10.13.2	Binomial series	257
10.13.3	Exponential and logarithmic series	258
10.13.4	Geometric and arithmetic series	258
10.13.5	Powers of natural numbers	258
10.13.6	Inverse natural numbers	259
10.13.7	Power series	259
10.13.8	Trigonometrical series	259
10.14	Some Well-known Inequalities	260
10.14.1	Harmonic mean-geometric mean-arithmetic mean inequality	260
10.14.2	Cauchy-Schwartz's inequality	260
10.14.3	Abel's inequality	260
10.14.4	Bernoulli's inequality and its generalizations	261
10.14.5	Chebyshev's inequality	261
10.14.6	Hölder's inequality	262
10.14.7	Minkowski's inequality	262
10.14.8	Convex function inequality	263
10.14.9	Samuelson's inequality	263
10.14.10	Inequalities on Mill's ratio	264
BIBLIOGRAPHY		265
GLOSSARY OF SOME FREQUENTLY USED ABBREVIATIONS AND SYMBOLS		291
INDEX		295

Chapter 1

MOMENTS, CUMULANTS, AND GENERATING FUNCTIONS

1.1 MOMENTS AND CUMULANTS

The moments and cumulants are a set of constants of a distribution which are useful for measuring its properties and, under certain circumstances, for specifying it.

1.1.1 DEFINITIONS

(1) *Expectation (One-Dimensional Case)*. Let X be a random variable and $h(\cdot)$ be a function with both domain and range the real line. The expectation or expected value of the function $h(\cdot)$ of the random variable X , denoted by $E[h(X)]$, is defined by

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) dx$$

if X is continuous with probability density function (pdf) $f(x)$ provided the integral converges absolutely (i.e., $\int_{-\infty}^{\infty} |h(x)| f(x) dx < \infty$); and by

$$E[h(X)] = \sum_x h(x)f(x)$$

if X is discrete with probability function (pf) $f(x)$ provided the sum converges absolutely (i.e., $\sum_x |h(x)|f(x) < \infty$). The summation is over all possible values of x .

If the integral or the sum in the above definition does not converge absolutely, we say $E[h(X)]$ does not exist.

(2) *Expectation (Two-Dimensional Case)*. Let (X, Y) be a bivariate random variable. The expected value of a function $h(\cdot, \cdot)$ of the bivariate random variable, denoted by $E[h(X, Y)]$, is defined to be

$$E[h(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f_{X, Y}(x, y) dx dy$$

if the random variable is continuous with pdf, $f_{X, Y}(x, y)$; and is defined to be

$$E[h(X, Y)] = \sum_x \sum_y h(x, y) f_{X, Y}(x, y)$$

if the random variable is discrete with pf $f_{X, Y}(x, y)$. The summation is over all possible values of (x, y) .

If the integral or the sum in the above definition does not converge absolutely, we say $E[h(X, Y)]$ does not exist.

(3) *Conditional Expectation*. Let (X, Y) be a bivariate random variable and $g(\cdot, \cdot)$ a function of two variables. The conditional expectation of $g(X, Y)$ given $X = x$, denoted by $E_{Y|X}[g(X, Y) | X = x]$, is defined to be

$$E_{Y|X}[g(X, Y) | X = x] = \int_{-\infty}^{\infty} g(x, y) f_{Y|X}(y|x) dy$$

if the random variable is continuous with $f_{Y|X}(y|x)$ as the conditional probability density function of Y given $X = x$ and

$$E_{Y|X}[g(X, Y) | X = x] = \sum_y g(x, y) f_{Y|X}(y|x)$$