Perspectives in Mathematical Logic

Keith J. Devlin

Constructibility

Keith J. Devlin
University of Lancaster
Department of Mathematics
Cartmel College
Bailrigg, Lancaster, LA1 4YL
England

AMS Subject Classification (1980):
03-02, 03E45, 03E05, 03E10, 03E25, 03E35, 04-02, 04A25, 04A30

ISBN 3-540-13258-9
Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-13258-9
Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging in Publication Data
Devlin, Keith J. – Constructibility.
(Perspectives in mathematical logic)
Bibliography: p. Includes index.
1. Constructibility (Set theory). I. Title. II. Series
QA248.D37 1984 511.3'22 84-5433
ISBN 0-387-13258-9 (U.S.)

Typesetting: Daten- und Lichtsatz-Service, Würzburg
Printing and binding: Graphischer Betrieb, Konrad Triltsch, Würzburg
2141/3140-543210

Preface to the Series

Perspectives in Mathematical Logic

(Edited by the Ω-group for "Mathematische Logik" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly since the mid-fifties, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. E. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and* *assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Oberwolfach, September 1975

Acknowledgements. *In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag and Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970 – 1973) as an initial help which made our existence as a working group possible.*

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F. K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. E. Sacks *D. S. Scott*

Author's Preface

This book is intended to give a fairly comprehensive account of the theory of constructible sets at an advanced level. The intended reader is a graduate mathematician with some knowledge of mathematical logic. In particular, we assume familiarity with the notions of formal languages, axiomatic theories in formal languages, logical deductions in such theories, and the interpretation of languages in structures. Practically any introductory text on mathematical logic will supply the necessary material. We also assume some familiarity with Zermelo-Fraenkel set theory up to the development or ordinal and cardinal numbers. Any number of texts would suffice here, for instance *Devlin* (1979) or *Levy* (1979).

The book is not intended to provide a complete coverage of the many and diverse applications of the methods of constructibility theory, rather the theory itself. Such applications as are given are there to motivate and to exemplify the theory.

The book is divided into two parts. Part A ("Elementary Theory") deals with the classical definition of the L_α-hierarchy of constructible sets. With some pruning, this part could be used as the basis of a graduate course on constructibility theory. Part B ("Advanced Theory") deals with the J_α-hierarchy and the Jensen "fine-structure theory".

Chapter I is basic to the entire book. The first seven or eight sections of this chapter should be familiar to the reader, and they are included primarily for completeness, and to fix the notation for the rest of the book. Sections 9 through 11 may well be new to the reader, and are fundamental to the entire development. Thus a typical lecture course based on the book would essentially commence with section 9 of Chapter I. After Chapter II, where the basic development of constructibility theory is given, the remaining chapters of Part A are largely independent, though it would be most unnatural to cover Chapter IV without first looking at Chapter III. Likewise, in Part B, after the initial chapter (Chapter VI) there is a large degree of independence between the chapters. (Indeed, given suitable introduction by an instructor, Chapter IX could be read directly after Chapter IV.)

Constructibility theory is plagued with a large number of extremely detailed and potentially tedious arguments, involving such matters as investigating the exact logical complexity of various notions of set theory. In order to try to strike a balance between the need to have a readable book of reasonable length, and the requirements of a beginning student of the field, as our development proceeds we give progressively less detailed arguments, relying instead upon the developing ability of the reader to fill in any necessary details. Thus the experienced reader may well find that it is necessary to skip over some of the earlier proofs, whilst the novice will increasingly need to spend time supplying various details. This is particularly true of Chapter II and the latter parts of Chapter I upon which Chapter II depends.

As this is intended as an advanced reference text, we have not provided an extensive selection of exercises. Those that are given consist largely of extensions or enlargements of the main development. Together with filling in various details in our account, these should suffice for a full understanding of the main material, which is their only purpose. The exercises occur at the end of each chapter (except for Chapter I), with an indication of the stage in the text which must be reached in order to attempt them.

Chapters are numbered by Roman numerals and results by normal numerals. A reference to "II.5" means section 5 of Chapter II, whilst "V.3.7" would refer to result 7 in section 3 of Chapter V. The mention of the chapter number would be supressed within that chapter. The end of a proof is indicated by the symbol □. If this occurs directly after the statement of a result, it should be understood that either the proof of the result is obvious (possibly in view of earlier remarks) or else (according to context) that the proof is a long one that will stretch over several pages and involve various lemmas. During the course of some of the longer proofs, many different symbols are introduced. In order to help the reader to keep track of them, at the points where new symbols are defined the symbol concerned appears in the outer margin of the book.

Finally, I would like to express my gratitude to all of those who have helped me in the preparation of this book. There are the members of the Ω-Group, who gave me the benefit of their views during the early stages of planning. Gert Müller kept a watchful eye on matters managerial, and Azriel Levy took on the task of editor, reading through various versions of the manuscript and making countless suggestions for improvements. Others who read through all or parts of the final manuscript are (in order of the number of errors picked up) Stevo Todorčević, Klaus Gloede, Jakub Jasinski, Włodek Bzyl, Martin Lewis, and Dieter Donder. Not to forget Ronald Jensen. Although he played no part in the writing of this book, it is clear (or will be if you get far enough into the book) that without his work there would have been practically nothing to write about!

Financial support during the preparation of the manuscript was provided by the Heidelberger Akademie der Wissenschaften.

Keith J. Devlin

Contents

Part A

Elementary Theory

Chapter I
Preliminaries

The fundamental set theory of this book is Zermelo-Fraenkel set theory. In this chapter we give a brief account of this theory, insofar as we need it. Sections 1 through 5 cover the early development of the theory up to ordinal and cardinal numbers. The remaining six sections deal with some special topics of direct relevance to the subject matter of this book, and the coverage is therefore a little more complete than in the previous sections.

1. The Language of Set Theory

The language of set theory, LST, is the first-order language with predicates $=$ (equality) and $\in$ (set membership), logical symbols $\wedge$ (and), $\neg$ (not), and $\exists$ (there exists), variables $v_0, v_1, \ldots,$ and (for convenience) brackets (,).

The *primitive* (or *atomic*) *formulas* of LST are strings of the forms

$$(v_m = v_n), \quad (v_m \in v_n).$$

The *formulas* of LST are generated from the primitive formulas by means of the following schemas: if Φ, Ψ are formulas, so too are the strings

$$(\Phi \wedge \Psi), \quad (\neg \Phi), \quad (\exists v_n \Phi).$$

(We generally use capital Greek letters to denote formulas of LST.)

The notions of *free* and *bound* variables are defined as usual. A *sentence* is a formula with no free variables.

We write $x \notin y$ for $\neg(x \in y)$ and $x \neq y$ for $\neg(x = y)$. (We generally use, x, y, z, etc. to denote arbitrary variables of LST.)

The defined logical symbols $\vee, \rightarrow, \leftrightarrow, \forall$ are introduced in the usual way, and are frequently treated as if they were basic symbols of LST (i.e. having the same status as $\wedge, \neg, \exists$). Likewise for the bounded quantifiers $(\exists v_m \in v_n)$ and $(\forall v_m \in v_n)$ (where $m \neq n$), introduced by the schemas:

$$(\exists v_m \in v_n) \Phi \quad \text{replaces} \quad \exists v_m ((v_m \in v_n) \wedge \Phi);$$

$$(\forall v_m \in v_n) \Phi \quad \text{replaces} \quad \forall v_m ((v_m \in v_n) \rightarrow \Phi).$$

The symbols $\subseteq$ and $\exists!$ are defined thus:

$$y \subseteq z \quad \text{abbreviates } (\forall x \in y)(x \in z);$$

$$\exists! x\Phi \quad \text{abbreviates } \exists y \forall x (y = x \leftrightarrow \Phi).$$

(Thus $\exists! x\Phi$ means "there is a unique x such that Φ".) We also write

$$y \subset z \quad \text{to mean } y \subseteq z \wedge y \neq z.$$

The above abbreviations are never regarded as a fundamental part of the language LST, however, unlike the bounded quantifiers, etc.

One final remark. In writing formulas, we strive for legibility at the expense of strict adherence to the syntax of LST. This particularly applies to our use of parentheses, which are omitted wherever possible. Also, when nesting of clauses is required, we sometimes use both (square) brackets as well as parentheses, for clarity. Out notation for the interpretation of variables in formulas is also chosen with clarity in mind. If we write, say, $\Phi(v_i, v_j)$, we mean that the free variables of Φ are amongst the variables v_1, v_j. If we subsequntly write $\Phi(x, y)$, where x and y are specific sets, we mean that Φ is a valid assertion when x interprets v_i and y interprets v_j. (Of course, we have also decided to use x, y, z, etc. to denote arbitrary variables of LST. But in any given case, the context should indicate the intended meaning.[1]

2. The Zermelo-Fraenkel Axioms

The theory ZF is the LST theory whose axioms are the usual axioms for first-order logic (for the langugage LST), together with the following axioms (i)–(vii):

(i) Extensionality: $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y)]$
(ii) Union: $\forall x \exists y \forall z [z \in y \leftrightarrow (\exists u \in x)(z \in u)]$
(iii) Infinity: $\exists x [\exists y (y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$
(iv) Power Set: $\forall x \exists y \forall z [z \in y \leftrightarrow z \subseteq x]$
(v) Foundation: $\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge (\forall z \in y)(z \notin x))]$
(vi) Comprehension (schema): $\forall \vec{a} \forall x \exists y \forall z [z \in y \leftrightarrow z \in x \wedge \Phi(z, \vec{a})]$,

where Φ is any LST formula whose free variables are amongst $z, \vec{a}$, and where the variables $\vec{a}, x, y, z$ are all distinct.

(We use $\vec{x}$, $\vec{a}$, etc. to denote finite strings of variables, $\forall \vec{a}$ to abbreviate $\forall a_1, \ldots, \forall a_n$ and $\Phi(z, \vec{a})$ to abbreviate $\Phi(z, a_1, \ldots, a_n)$. In more complicated situations,

1 Strictly speaking there is no clash of notation here. As far as formal set theory is concerned there are simply *variables* (to denote "sets"). But as usual, to avoid incomprehensible use of quantifiers and formulas to define specific sets, we argue in a loose, *semantic* fashion whenever possible, and then it can be useful to distinguish between "formal variables" and "sets which interpret those variables".

we often use expressions such as $\vec{x}_0, \dots, \vec{x}_n$. Here, $\vec{x}_0$ will denote some sequence $x_{00}, \dots, x_{0k}$, $\vec{x}_1$ will be another sequence $x_{10}, \dots, x_{1l}$, possibly of a different length, according to context, and so on.)

(vii) Collection (schema):

$$\forall \vec{a} [\forall x \exists y \Phi(y, x, \vec{a}) \rightarrow \forall u \exists v (\forall x \in u)(\exists y \in v) \Phi(y, x, \vec{a})],$$

where Φ is any LST-formula whose free variables are amongst $y, x, \vec{a}$, and where the variables $\vec{a}, x, y, u, v$ are all distinct.

In (iii), the exact formulation of the Axiom of Infinity is not important, and different texts often give different formulations. The main point is to guarantee the existence of at least one infinite set. Axiom (vi) (the Comprehension Axiom schema) is sometimes referred to as the Subset Selection schema. The German word *Aussonderungsaxiom* is also quite common for this axiom scheme. In Axiom (vii) (Collection), notice that we have placed the variable y before the variable x. This is purely a stylistic convention, of course, and reflects the fact that in our representation of a function as a set of ordered pairs, we shall take the first member of each ordered pair as the value of the function and the second element as the argument. Axiom schemas (vi) and (vii) are often replaced by a single schema: the *Axiom of Replacement*.

Notice that by virtue of the two axiom schemas, the above list of axioms for ZF is infinite. We shall soon be able to prove that no finite collection of LST sentences suffices to axiomatise ZF.

By the Axiom of Infinity, there exists at least one set. The Axiom of Comprehension then yields the existence of the empty set $\emptyset$. Many texts include as an axiom of ZF the *Null Set Axiom*, which is the assertion that there exists a set having no elements, viz.:

$$\exists x \forall y (y \notin x).$$

Zermelo-Fraenkel set theory includes one further axiom:

(viii) Axiom of Choice (AC):

$$\forall x [(\forall y \in x)(y \neq \emptyset) \wedge (\forall y, y' \in x)(y \neq y' \rightarrow \forall w (w \in y \leftrightarrow w \notin y') \rightarrow (\exists z)(\forall y \in x)(\exists! v \in y)(v \in z)].$$

We denote Zermelo-Fraenkel set theory (which includes AC) by ZFC. This nomenclature is now fairly standard, despite the rather unfortunate fact that it means that the letters ZF do not stand for "Zermelo-Fraenkel" set theory, but just a part of that theory. To try to avoid any confusion, throughout the book we shall stick to the abbreviated notations ZF and ZFC. Hence, we shall have the "equation"

$$\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}.$$

ZFC is our basic set theory. On occasions it will be important to note that AC is not being used in an argument, and in such cases we shall write, for example,

$$\mathrm{ZF} \vdash \Phi$$

or else

$$\Phi \to_{\mathrm{ZF}} \Psi$$

to mean, respectively, that Φ is provable in ZF or that Ψ is provable from Φ together with the axioms of ZF.

3. *Elementary Theory of ZFC*

3.1 (Sets and Classes). The basic objects of discussion of ZFC (i.e. the objects over which the variables range) are called *sets*. The *universe* is the collection of all sets, and is denoted by V. If $\Phi(v_0, v_1, \ldots, v_n)$ is an LST formula and $x_1, \ldots, x_n$ are sets, the collection of all sets x for which $\Phi(x, x_1, \ldots, x_n)$ is a *class*, denoted by

$$\{x \mid \Phi(x, x_1, \ldots, x_n)\}.$$

Every set, y, is a class (consider the formula $\Phi(x, y) \equiv (x \in y)$), but not every class is a set (consider the formula $\Phi(x) \equiv (x \notin x)$, which would lead at once to the Russell paradox if the class it defined were a set). We often write

$$\{x \in y \mid \Phi(x, x_1, \ldots, x_n)\}$$

in place of

$$\{x \mid x \in y \wedge \Phi(x, x_1, \ldots, x_n)\}.$$

(By the Axiom of Comprehension, this class is always a set.) We generally use capital Roman letters X, Y, Z etc. to denote classes, with lower case Roman letters being reserved for sets (as well as for variables of LST, which denote sets, of course). A class which is not a set is called a *proper class*. Proper classes do not fall under the scope of the axioms of ZFC, but their usage is convenient. We assume the reader is familiar both with the use of proper classes in set theory and the means by which such usage may be avoided if required. A particular example occurs in VI.1, where we discuss the rudimentary functions. It is convenient, though avoidable, to develop the relevant theory in terms of "functions" defined on the whole of V, even though, as proper classes these cannot be *functions* in the sense of set theory at all.

Our set-theoretic notation is standard. The set consisting of precisely the elements $x_1, \ldots, x_n$ is denoted by

$$\{x_1, \ldots, x_n\},$$

$\{x\}$ is the *singleton* of x, and $\{x, y\}$ is the *unordered pair* of x, y. Many texts include as an axiom of ZF the *Pairing Axiom*, which asserts that for every pair of elements x, y, the set $\{x, y\}$ exists, i.e.

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u = x \vee u = y).$$

However, as this "axiom" is easily proved from the axioms we listed earlier, we did not take it as a basic axiom.

The *ordered pair* of x and y is defined by

$$(x, y) = \{\{x\}, \{x, y\}\},$$

and has the property that

$$(x, y) = (x', y') \quad \text{iff } x = x' \text{ and } y = y'.$$

The *union* of x (i.e. the set of all members of all members of x) is denoted by $\bigcup x$, and is guaranteed to exist by the Union Axiom. We write $x \cup y$ instead of $\bigcup \{x, y\}$. The *intersection* of x, $\bigcap x$, is defined by

$$y \in \bigcap x \quad \text{iff } (\forall z \in x)(y \in z),$$

and is a set whenever $x \neq \emptyset$. (By our definition, $\bigcap \emptyset = V$, but this is not a case that will ever concern us.) We write $x \cap y$ for $\bigcap \{x, y\}$. The *difference* of x and y is defined by

$$x - y = \{z \in x \mid z \notin y\}.$$

The *power set* of x (i.e. the set of all subsets of x) is denoted by $\mathscr{P}(x)$, and is guaranteed to exist by the Power Set Axiom.

3.2 (Ordinals). A class M is said to be *transitive* if

$$x \in y \in M \rightarrow x \in M.$$

If $\text{Trans}(v_0)$ denotes the LST formula

$$(\forall v_1 \in v_0)(\forall v_2 \in v_1)(v_2 \in v_0),$$

then a set x will be transitive iff $\text{Trans}(x)$.

An *ordinal number* (or simply, an *ordinal*) is a transitive set which is linearly ordered by $\in$. We use $\alpha, \beta, \gamma, \ldots$ to denote ordinals. We denote by $\text{On}(v_0)$ the LST-formula

$$\text{Trans}(v_0) \wedge (\forall v_1 \in v_0)(\forall v_2 \in v_0)(v_1 = v_2 \vee v_1 \in v_2 \vee v_2 \in v_1).$$

It is not hard to show that a set x will be an ordinal iff $\text{On}(x)$.

If α, β are ordinals, either $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$. So the class

$$\mathrm{On} = \{x \mid \mathrm{On}(x)\}$$

is totally ordered by $\in$. We often write $\alpha < \beta$ instead of $\alpha \in \beta$, and $\alpha \leqslant \beta$ instead of $(\alpha < \beta \vee \alpha = \beta)$. It is easily seen that $\alpha < \beta$ is equivalent to $\alpha \subset \beta$. Moreover, for any ordinal α,

$$\alpha = \{\beta \mid \beta < \alpha\}.$$

By the Axiom of Foundation, the relation $<$ is in fact a well-ordering of On (i.e. every non-empty subset of On has a $<$-least element).

If A is a set of ordinals, then $\cup A$ is also an ordinal. In fact, $\cup A$ is the least ordinal δ such that $(\forall \alpha \in A)(\alpha \leqslant \delta)$. This least δ is also called the *supremum* of A, denoted by $\sup(A)$. Thus $\sup(A)$ and $\cup A$ coincide.

The first ordinal (under the canonical well-ordering $\in$) is the null set, $\emptyset$, but when considered as an ordinal it is usually denoted by 0. The next ordinal is the set $\{0\}$, denoted by 1. Then comes the ordinal $\{0, 1\}$, denoted by 2, followed by $3 = \{0, 1, 2\}$, and so on. If α is an ordinal, so too is $\alpha \cup \{\alpha\}$, and there is no ordinal γ strictly between α and $\alpha \cup \{\alpha\}$. We call $\alpha \cup \{\alpha\}$ the *successor* of α, denoted by $\alpha + 1$. Any ordinal of the form $\alpha + 1$ is called a *successor ordinal*. An ordinal α is a successor ordinal iff $\mathrm{succ}(\alpha)$, where $\mathrm{succ}(v_0)$ is the LST-formula

$$\mathrm{On}(v_0) \wedge (\exists v_1 \in v_0)(\forall v_2 \in v_0)(v_2 \in v_1 \vee v_2 = v_1).$$

A non-zero ordinal which is not a successor ordinal is called a *limit ordinal*. If $\lim(v_0)$ is the LST-formula

$$\mathrm{On}(v_0) \wedge (\exists v_1 \in v_0)(v_1 = v_1) \wedge (\forall v_1 \in v_0)(\exists v_2 \in v_0)(v_1 \in v_2),$$

then an ordinal α will be a limit ordinal iff $\lim(\alpha)$. Using the Axiom of Infinity, together with other ZF axioms, it can be shown that a limit ordinal exists. The least limit ordinal is denoted by ω. The elements of the set ω are precisely the finite ordinal numbers, and are called the *natural numbers*. We usually denote natural numbers by m, n, i, j, k, etc. Notice that ω is definable by the formula

$$\lim(v_0) \wedge (\forall v_1 \in v_0)(\mathrm{succ}(v_1) \vee (\forall v_2 \in v_1)(v_2 \neq v_2)).$$

We usually write $\exists \alpha \Phi(\alpha)$ in place of

$$\exists v_0 [\mathrm{On}(v_0) \wedge \Phi(v_0)],$$

and $\forall \alpha \Phi(\alpha)$ in place of

$$\forall v_0 [\mathrm{On}(v_0) \rightarrow \Phi(v_0)].$$

If $(X, <)$ is a well-ordered set, there is a unique ordinal number α such that $(X, <)$ is isomorphic to α (with the usual ordering). This α is called the *order-type* of $(X, <)$, denoted by $\mathrm{otp}(X, <)$.