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GENERALIZED CLASSICAL MECHANICS AND FIELD THEORY

A Geometrical Approach of Lagrangian and Hamiltonian Formalisms Involving Higher Order Derivatives

Manuel DE LEON

and

Paulo R. RODRIGUES

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Manuel DE LEÓN

Departamento de Geometria y Topologia
Facultad de Matemáticas
Universidad de Santiago de Compostela
España.

and

Paulo R. RODRIGUES

Departamento de Geometria
Instituto de Matemática
Universidade Federal Fluminense
Niterói, RJ
Brasil

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"Il est peut-être inutile de faire observer qu'il n'est point de Mathématiques "sans larmes" à l'usage exclusif des physiciens et que, si je me suis efforcé de choisir et coordonner celles des théories mathématiques qui peuvent être utiles aux physiciens, il m'était impossible de renoncer dans leur exposé à cette rigueur sans laquelle il n'est plus de science ni mathématique ni physique."

A. Lichnerowicz.

(Algèbre et Analyse Linéaires, Masson, Paris, (1948)).

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GENERALIZED CLASSICAL MECHANICS AND FIELD THEORY

FOREWORD

The aim of this book is to build up a large panel of the present situation of Lagrangian and Hamiltonian formalisms involving higher order derivatives. The achievements of Differential Geometry in formulating a more modern and powerful treatment of these theories are developed, including the contributions of the author's themselves. An extensive review of the development of these theories in classical language is also given.

A Lagrangian formalism is said to be of <u>higher order derivatives</u> if it is described by a real (smooth) function L which depends on n-independent variables x_a, m-functions $y^A(x_a)$ and all derivatives of the y's with respect to the x's up to a certain finite order k. For sake of simplicity, we will say that L is a <u>Lagrangian of order</u> k. In Particle Mechanics and Field Theories one usually works with Lagrangians of order one. Therefore, higher order Particle Mechanics (resp. Field Theories) means that the Lagrangians depend not only on position/velocity variables (resp. independent coordinates/field variables) but also on their time derivatives up to k-th order (resp. partial derivatives up to k-th order of the field variables with respect to the independent coordinates).

Higher order Hamiltonian formalisms will understand, as in the standard theory of order one, the Hamiltonian counterpart of such Lagrangians.

There is not much agreement in the literature as to the interest in this kind of problem. It seems to have started with M. Ostrogradsky in 1848 (see Whittaker (1959)). According to P. Dedecker (1979), it was J. Jacobi who first studied these systems. Also Todhunter, in his book "History of the Calculus of Variations", mentions that Clebsch, following a suggestion of Jacobi, considered this subject in 1858. Therefore, we may call such theory <u>Jacobi-Ostrogradsky Generalized Classical Mechanics</u> (and Field Theory) or, more simply, <u>Generalized Classical Mechanics</u> (and Field Theory).

In the last 40 years many papers dealing with higher derivatives in Mechanics and Field Theories have appeared. It seems that it was F. Bopp (1940) and B. Podolsky (1942) who renewed the interest in this kind of generalization in physics. Podolsky (and co-workers), for example, introduced an electromagnetic theory with second order derivatives.

Mechanics lays naturally in Differential Geometry and reciprocally. This "marriage" has allowed not only a more rigorous formulation from the mathematical point of view, but also a better understanding of its physical content. Following the results in Symplectic Mechanics systematized in the literature (see the "Bibles": Abraham & Marsden (1978), Arnol'd (1974) and Godbillon (1969)) it is found that a Lagrangian, resp. Hamiltonian, formalism can be characterized by geometric structures canonically associated to the tangent, resp. cotangent, bundle of a given differentiable manifold (they are respectively, the velocity, phase and configuration spaces). Analytical Mechanics is developed through geometric formalisms underlying the theory of fiber bundles.

These last years a certain number of papers where the geometrical formulation of this so-called Generalized Mechanics and Field Theory is developed have been published. Inspired by them, the major task of the present work will be to give an overview of the results obtained in the elaboration of this formalism, including our own results on the subject. The "natural place" of our study will be the Jet bundles, first introduced by Ch. Ehresmann in the years of 1950. For instance, in Lagrangian Particle Mechanics we develop the formalism on tangent bundles of higher order. We will emphasize some geometric structures underlying such a Mechanics in the sense that they are a generalization of the methods usually employed in the standard situation.

The text is divised in three chapters, each of them with an introduction. In the first we give some geometric tools necessary for the development of the others two chapters. In Chapter II we adopt the point of view of J. Klein (1962) for Lagrangian Mechanical Systems. Klein's Lagrangian formalism is developed with the help of the Almost Tangent Geometry (introduced by Clark & Bruckheimer in 1960) and a special exterior differential calculus. We extend this geometry to higher order tangent bundles. One advantage of such a choice is that it is possible to give an intrinsical exposition without carrying the symplectic form of the cotangent bundle, as we usually do for the standard regular Lagrangians. Therefore, we work, generically, with pre-symplectic structures in the place of symplectic ones. The third chapter is devoted to a local and sometimes global study of Generalized Classical Field Theory from the variational approach. The geometric formalism adopted there is the usual one underlying exterior differential calculus on manifolds.

This book is addressed mainly to graduate students. Of course it is assumed that the readers are acquainted with the geometrical formulation of standard Classical Mechanics.

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The authors.

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CHAPTER I

THE DIFFERENTIAL GEOMETRY OF HIGHER ORDER JETS AND TANGENT BUNDLES

I.1. - Introduction

This chapter is devoted to the study of basic geometrical notions required for the development of the main object of the text. Some facts about Jet theory are reviewed in Section 2, and the reader may also consult the articles of Ehresmann (1951), (1954), (1955)(a), (1955)(b), the appendix of Aldaya & Azcárraga (1980), as well as the books of Golubitsky & Guillemin (1973) and Michor (1980).

In Section 3, a particular case of Jet manifolds is considered: the tangent bundle of higher order. We show that this jet bundle posseses in a canonical way a certain kind of geometric structure, the so called almost tangent structure of higher order, introduced by Eliopoulos (1966), and which is a generalization of the almost tangent geometry of the tangent bundle. This almost tangent geometry of higher order provides a special differential calculus which is a generalization of the formalism presented in the last chapters of the book of Godbillon (1969) on Differential Geometry and Classical Mechanics.

Another important fact examined in this chapter is the extension of the notion of "spray" to higher order tangent bundles. This concept was introduced for the ordinary situation by Ambrose, Palais & Singer (1960), and it is relevant in Mechanics. The theory of sprays is closely related to the theory of connections on manifolds; therefore, connections of higher order are introduced and the relation between sprays and connections is studied in some detail.

I.2. - Jet manifolds

2.1. Jets of sections

It is assumed throughout the text that all structures, mappings, etc., are smooth (C^∞-class). Let N and M be manifolds with $\dim N = n$.

DEFINITION (1). Consider the triple (M,p,N), where $p: M \to N$ is a mapping. We say that M is a fibered manifold over N with projection p if the following conditions are verified:

(i) $\dim M = n+m$, where m is a positive integer,

(ii) p is a surjective submersion,

(iii) for any point in M there are two charts (U,f) and (V,g) of M and N, respectively, with $p(U) = V$,

such that $p_1 \circ f = g \circ p$, where $p_1: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the canonical projection onto $\mathbb{R}^n$.

REMARK (1). We can easily check that conditions (i) and (ii) are a direct consequence of condition (iii).

A mapping $s: N \to M$ is said to be a section of M if $p \circ s = Id_N$ (identity on N) and is said a local section if $p \circ s/U = Id_U$, for an open subset U of N. We put $Sec(M)$ f the set of all sections on M and $\mathrm{Sec}_U(M)$ for local sections (sometimes we say "s is a section along U").

Let (M,p,N) be a fibered manifold and F a real vector space of finite dimension. Suppose that

(i) for each $a \in N$, the fiber $N_a = p^{-1}(a)$ admits a vector space structure isomorphic to F;

(ii) there is an open covering $\{U_i\}_{i\in I}$ of N so that for each $i \in I$ there exists a diffeomorphism $f_i\colon p^{-1}(U_i) \to U_i \times F$ such that for each $a \in U_i$ the restriction $f_i|_{N_a}$ is an isomorphism from N_a to $\{a\}\times F$; then (M,p,N) is said to be a (locally trivial) <u>fibered vector bundle</u>, or, simply, vector bundle (for more indications on vector bundles see Appendix A).

If $p\colon M \to N$ is a fibered manifold we shall denote by $\mathrm{Ver}\, M$ the <u>vertical bundle</u> of M over N, that is, $\mathrm{Ver}\, M$ is the vector subbundle of TM consisting of all tangent vectors to M which are projected onto 0 by Tp.

DEFINITION (2). Way say that $s,s' \in \mathrm{Sec}(M)$ are $\sim_k$-<u>related</u>, $0 \le k \le \infty$, in a point $x \in N$, if

(i) $s(x) = s'(x)$;

(ii) for all functions $f\colon M \to \mathbb{R}$, the function $f\circ s - f\circ s'\colon N \to \mathbb{R}$ is "flat" of order k at x, that is, this function and all their derivatives up to order k, included, are zero at x.

DEFINITION (3). The equivalence class determined by $\sim_k$ is called <u>jet of order</u> k, or, simply, k-<u>jet</u> at x. For $s \in \mathrm{Sec}(M)$, the k-jet of s at x is represented by $j^k s(x)$,

$j^k_x s$ or $\tilde{s}^k(x)$. The set of all k-jets at x is denoted by $J^k_x(M,p,N)$. We put $J^k(M,p,N)$ for the union $\bigcup_x J^k_x(M,p,N)$. When the fibered manifold is the same throughout the text we put simply $J^k M$ or, else, J^k.

It is possible to define jets for local sections. In such a case, it is simpler to work with germs of sections. A <u>germ</u> of a section is the equivalence class determined by the relation: two sections are related if they have the same value at every point in the intersection of their domains.

2.2. Jets of mappings

Let N and M be manifolds. In a similar way as we did for sections, we may consider a more general situation defining the notion of k-jet at a point for a mapping from N to M. If $f: N \to M$, then the equivalence class determined by $\sim_k$ is called the k-jet of f at x. We put also $j^k f(x)$, $j^k_x f$ or $\tilde{f}^k(x)$ for a representative of the class. The set of k-jets at x is now represented by $J^k_x(N,M)$ and, also, $J^k(N,M) = \bigcup_x J^k_x(N,M)$. In a similar way, we can define k-jets for mappings defined locally on N (and so, we may consider k-jets of their germs).

REMARK (2). We say that a fibered manifold (M,p,N) is <u>trivial</u> if there is a manifold B such that $M = N\times B$. Since the graph of any map $f: N \to B$ is the corresponding section of the fibered manifold $N\times B$ over N, we can speak equivalently of a map from N to B or of a section of $N\times B$ over N. Throughout the text we will consider only this situation and we shall dentify $J^k(N\times B,p,N)$ with $J^k(N,B)$.

It can be shown that the set of all k-jets of sections (or maps) admits a (smooth in our case) differentiable structure (see Pommaret (1978) or Golubitsky & Guillemin (1973), for example).

The k-jet manifold of mappings (or sections) can be fibered in different ways:

α^k: $J^k(N,M) \to N$; $\alpha^k(\tilde{f}^k(x)) = x$ (called <u>source</u> projection)

β^k: $J^k(N,M) \to M$; $\beta^k(\tilde{f}^k(x)) = f(x)$ (called <u>target</u> projection)

ρ^k_r: $J^k(N,M) \to J^r(N,M)$; $\rho^k_r(\tilde{f}^k(x)) = \tilde{f}^r(x)$, where $r \leq k$.

The manifold $J^k(M,p,N)$ is a submanifold of $J^k(N,M)$ and so the above projections admit restrictions to it (denoted also by the same symbol). $(J^k(M,p,N), \alpha^k,N)$, $(J^k(M,p,N), \beta^k,M)$, etc., are the corresponding fibered manifolds. It is clear that $\alpha^k = p\circ\beta^k$ and $\alpha^r\circ\rho^k_r = \alpha^k$, where $r \leq k$.

2.3. <u>The k-jet prolongation</u>

Let N and M be manifolds. (M may be fibered over N). Let J^kM be the manifold of k-jets (for sake of symplicity, we identify here both notations).

DEFINITION (4). The mapping which associates every point $x \in N$ to the k-jet of a mapping $g: N \to M$ at x is called the k-<u>jet prolongation</u> (or <u>extension</u>) of g and is represented by j^kg or $\tilde{g}^k$.

So $\tilde{g}^k: N \to J^kM$ is defined by $x \to \tilde{g}^k(x)$. It is clear that $\tilde{g}^k$ is a section of the fibered manifold (J^kM,α^k,N)

(the same considerations are true for the local situation). It is also clear that $\beta^k \circ \tilde{g}^k = g$.

Let us consider the fibered manifolds (M,p,N) and (J^kM,α^k,N). If $u: N \to J^kM$ is a section, then, in general, only locally there exists a section $s: N \to M$ such that $\tilde{s}^k = u$.

2.4. A particular situation

Let $N = \mathbb{R}^n$ and $M = \mathbb{R}^n \times \mathbb{R}^m$. We identify the sections of such (trivial) manifold to the mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ as well as their k-jets. If $f: \mathbb{R}^n \to \mathbb{R}^m$ is repreented by

$$f(x) = (f^1(x_1,\dots,x_n),\dots,f^m(x_1,\dots,x_n)),$$

then, for sake of simplicity, we put $f(x) = (f^A(x_a))$, with $1 \leq a \leq n$, $1 \leq A \leq m$.

PROPOSITION (1). Two mappings f,g from $\mathbb{R}^n$ to $\mathbb{R}^m$ have the same k-jet at x in $\mathbb{R}^n$ if and only if for every A in $(1,\dots,m)$, f^A and g^A have the same Taylor polynomial expansion at x, truncated at order k (inclusive).

Proof: Suppose that f^A and g^A have the same Taylor expansion of order k at x. Then $f(x) = g(x)$ and, if $h: \mathbb{R}^m \to \mathbb{R}$ is an arbitrary function, then the Taylor's expansion of $h \circ f$ and $h \circ g$ at x is obtained by the substitution of Taylor's expansion of f^A and g^A (at x) into the Taylor expansion of h at $f(x)$ and $g(x)$. So $h \circ f - h \circ g$ is flat or order k at x. The converse is trivial and so the proposition is proved. □

REMARK (3). It follows that all k-jets at $x \in \mathbb{R}^n$ can be identified to the m-tuples of polynomials on n variables with degree k. If $J^k(n,m)$ is the vector space of such polynomials, excluded the constant terms, then $J^k(\mathbb{R}^n,\mathbb{R}^m)$ can be identified to $\mathbb{R}^n \times \mathbb{R}^m \times J^k(n,m)$ (and, now, it is clear how one obtains the dimension of such manifold, given by (*) in §2.1, Ch. III). If $P^k f(x)$ denotes the Taylor expansion of f at x up to order k, then $\tilde{f}^k(x) \rightsquigarrow (x, f(x), P^k f(x) - f(x))$ gives the mentioned identification.

Let us see a very simple example: if $f(x) = x^3$, identifying $J^3(1,1)$ to $\mathbb{R}^3$ with the aid of $at+bt^2+ct^3 \rightsquigarrow$ $\rightsquigarrow (a,b,c)$, then the Taylor expansion of $f(x+t)$ up to order 3 is $x^3+3x^2t+3xt^2+t^3$ and so

$$\tilde{f}^3(x) = (x, f(x), P^3 f(x) - f(x)) = (x, x^3, 3x^2, 3x, 1).$$

Generally, the term of order r of the Taylor polynomial is

$$\frac{1}{r!}\left(\sum_1^m t_a \frac{\partial}{\partial x_a}\right)^r f(x) = \frac{1}{r!}\sum_1^m {}_{a_1,\dots,a_r} t_{a_1\dots a_r} \frac{\partial^r f}{\partial x_{a_1}\dots\partial x_{a_r}}(x),$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth, $x \in \mathbb{R}^n$ and $1 \le a_1 \le \dots \le a_r \le n$.

If we put $f(x) = (f^A(x_a)) = (y^A(x_a))$ and $y^A_{a_1\dots a_r} =$ $= (\partial^r y^A/\partial x_{a_1}\dots\partial x_{a_r})$, then the k-jet at x of f is represented by

$$\tilde{f}^k(x) = (x_a, y^A, y^A_{a_1\dots a_r}). \tag{1}$$

To simplify more the notation we shall conventionate that

$$a(r) = a_1\dots a_r .$$

For a summation index: