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Erwin Bolthausen
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# Spin Glasses

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# Preface

Spin glasses have become a paradigm for highly complex disordered systems. In the 1960'ies, certain magnetic alloys were found to have rather anomalous magnetic and thermal properties that seemed to indicate the existence of a new kind of phase transition, clearly distinct from conventional ferromagnetic materials. The origin of these anomalies was soon deemed to lie in two features: the presence of competing signs in the two-body interactions, and the disorder in the positions of the magnetic atoms in the alloy. This has led to the modelling of such materials in the form of spin-systems with random interactions. In the 1970ies, two principle models were proposed: the Edwards-Anderson model, which is a lattice spin system with random nearest neighbor interactions and as such is the randomized version of the classical Ising model; and the Sherrington-Kirkpatrick model, proposed as a mean field model, where all spins interact with each other on equal footing, which is a randomized version of the Curie-Weiss model. The SK-model was clearly intended to provide a simple, solvable caricature of the Edwards-Anderson model, that should give some insights into the nature of the spin glass transitions, just as the Curie-Weiss model allows a partial understanding of ferromagnetic phase transitions. The remarkable interest that the spin glass problem has received is largely due to the fact that neither of the two models turned out to be easily tractable. The Sherrington-Kirkpatrick model was solved on a heuristic level through the remarkable "replica symmetry breaking" ansatz of Parisi, which not only involved rather unconventional mathematical concepts, but also exhibited that the thermodynamic limit of this model should be described by an extraordinarily complex structure. The short-range Edwards Anderson model has been even more elusive, and beyond some rather rudimentary rigorous results, most of our insight into the model is based on numerical simulations, which in themselves prove to be a highly challenging task.

Mathematicians became interested in this problem in the late 1980ies, but on a larger scale in the 1990ies, starting with work of Pastur and Shcherbina, and the systematic programmes initiated by Guerra on the one hand and Talagrand on the other. In 1996 a workshop in Berlin brought together the leading experts in the field. The state of the art at that time is to a large extend documented in the volume "Mathematical Aspects of Spin Glasses and Neural Networks", edited by A. Bovier and P. Picco (Birkhäuser, 1997). Since then, the progress made in the field has exceeded all expectations. Even as we began planning for a new workshop on the mathematics of spin glasses that was finally held at the Centro Stefano Franscini on the Monte Verità, we did not anticipate that he timing of the event would allow to present for the first time some ground breaking progress. In 2002, Francesco Guerra published an upper bound on the free energy of the SK model that coincided with the Parisi solution. This was the first time that this remarkable construction was to be related to a mathematically rigorous result. Less than a year later, Michel Talagrand announced that he could prove the corresponding lower bound, thus establishing the Parisi solution in a fully rigorous manner.

The Monte Verità meeting thus fell into a most exciting period. It was attended by most of the leading experts on spin glasses, including David Sherrington, Giorgio Parisi, Francesco Guerra, Michel Talagrand, Michael Aizenman, Chuck Newman, and Daniel Stein, to name a few. Besides the reports on the progress mentioned above, the participants and invited speakers reported on a wealth of interesting new results around spin glasses, both on the static and dynamic aspect. As a result we decided to collect a number of invited review papers to document the state of the art in spin glass theory today. The result of this is the present book. It contains a general introduction to the spin glass problem, written by E. Bolthausen, that will serve in particular as a pedagogical guide to the description of the nature of the Parisi solution and the derivation of Guerra's bound in the formulation of Aizenman, Sims, and Star. A. Bovier and I. Kurkova shed light on the Parisi solution from another angle by deriving and describing the asymptotics of the Gibbs measure in another class of spin glass models, the Generalized Random Energy models, in full detail. D. Sherrington gives an account of the history of the spin glass problem from a more physical perspective. M. Talagrand's contribution is a pedagogical presentation of his celebrated proof of the validity of the Parisi solution. Two articles by Ch. Newman and D. Stein discuss the latest developments in the ongoing dispute on the question, whether the predictions of the mean field Sherrington-Kirkpatrick model have any implications for the behavior of short range spin glasses. Finally, A. Guionnet gives an account of what has been achieved in the understanding of another outstanding issue about spin glasses, namely their non-equilibrium properties.

We hope that this volume will serve as a reference handbook for anyone wanting to get an idea of where we are in the theory of spin glasses, and what this subject is all about.

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# Contents

# Random Media and Spin Glasses: An Introduction into Some Mathematical Results and Problems

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## 1 Introduction

No materials in the history of solid state physics have been as intriguing and perplexing than certain alloys of ferromagnets and conductors, such as AuFe or CuMg, known as *spin glasses*. The attempts to model these systems have led to a class of *disordered spin systems* whose mathematical analysis has proven to be among the most fascinating fields of statistical mechanics over the last 25 years. Even the seemingly most simple model class, the *mean-field models* introduced by Sherrington and Kirkpatrick [1] now known as SK-models have proven to represent an amazingly rich structure that is mathematically extraordinarily hard to grasp. Theoretical physics has produced an astounding solution describing the thermodynamics properties of these models that is based on ad hoc constructions (so-called "replica symmetry breaking" [2]) that so far have largely resisted attempts to be given a concrete mathematical sense. From a purely mathematical point of view, the problem posed here represents a canonical problem in the theory of stochastic processes in high dimensions and as such the interest in it transcends largely the original physical question. The fact that the heuristic approach of theoretical physics, if given a clear mathematical meaning, would give a totally new and powerful tool for the analysis of such questions is the reason why there has been a strong upsurge in interest from within the mathematical, and in particular probabilistic community in this and related problems. Moreover, the same types of mathematical problems arise in many areas of applications that are of great current interest. For example, heuristic methods of statistical mechanics make powerful prediction concerning numerous problems of combinatorial optimization, computer science, and information technology.

For a long time progress on mathematically rigorous results in this field have been extremely limited, but over the last years the situation has changed considerably due to the results of Bovier, Comets, Derrida, Gayrard, Newman, Pastur, Picco, Shcherbina, Stein, Talagrand, and Toninelli, for instance. Michel Talagrand has developed in a systematic way an induction technique known as the "cavity method" as a tool to analyze in a rigorous way random Gibbs measures. This has allowed him to confirm predictions made by the heuristic "replica method" mostly in domains where the so-called replica-symmetric solution is predicted to hold; in mathematical terms, this corresponds to situations where the Gibbs measure is asymptotically a (random) product measure. The cavity method then allows to precisely compute the corresponding parameters. Interestingly, the method can also be applied in some situations where the Gibbs measure is a nontrivial mixture of product measures ("one-step replica symmetry breaking"), such as the $p$-spin Sherrington–Kirkpatrick model. Much of this can be found in Talagrand's book [3].

Another discovery was made by Guerra and Ghirlanda. This concerns a set of recursive relations between so-called multioverlap distributions. In certain cases it could be shown that they determine a universal structure in the Gibbs measures of these systems. In particular, these identities proved crucial in the work of Talagrand on the $p$-spin models, and in recent work of Bovier and Kurkova who used them to prove convergence and describe the limit of the Gibbs measures in a class of models introduced by Derrida, the so-called generalized random energy models.

The most spectacular successes recently, initiated by Francesco Guerra, are coming from interpolation techniques between different processes. Such methods are in principle well established in the analysis of Gaussian processes. Nonetheless, their judicious use has led to very remarkable results: Guerra and Toninelli [4] used them to prove the existence of the limit of the free energy in the SK (and many similar) models. A bit later, Guerra [5] has been able to prove that the predicted expression for the free energy of the SK-model from replica theory is at least a lower bound, and finally, Talagrand [6] has been able to refine the technique and combine it with his cavity method to control the error in Guerra's bound, and in this way he proved the Parisi formula [7] in the full temperature regime of the SK-model. Despite of these successes, there still remain many open problems, and it is perhaps fair to say that even the SK-model, where the Parisi formula has now been proved, is still very poorly understood. For instance, an understanding of the so-called ultrametricity is completely lacking, although it is at the very heart of the physics theory of the model. Even more importantly, there are many models where interpolation techniques had been far less successful, and where our understanding is till restricted to the part of the parameter space outside the spin glass phase.

In a second development, the analysis of the stochastic dynamics of highly disordered model is starting to make progress. Important contributions are due to Ben Arous and Guionnet and Grunwald, who derived asymptotic dynamics in Langevin and Glauber dynamics of the SK-model. Spin glass dynamics is supposed to have so-called "aging" which means that the systems react slower the older it gets. There are mathematically precise descriptions of this behavior which, however, have not yet been proved for the SK-model, but for simpler models there has been a lot of progress, recently (see the paper of Alice Guionnet in this volume).

In this introductory notes, I will give an overview for some of the developments, but I will mainly focus on mathematical results, and on results which are developed later in other contributions in this volume in more details.

For an overview over recent developments and perspectives in physics, see the article of Sherrington [8] in this volume. A topic which I leave out in this introduction is short-range spin glass models. This is presently still quite a controversial subject, even in the physics literature. In recent years, Newman and Stein [9, 10] have obtained results.

The focus given here in my introductory notes is on the mean-field model, and in particular on the recent mathematical developments around Guerra's interpolation technique (Sects. 4.1 and 5.3), the Talagrand's version of the cavity method in Sect. 4.2, and the random energy models in Sect. 5. For more on this subject, see the article by Bovier and Kurkova [11]. For a more in depth overview of Talagrand's application of the Guerra interpolation to SK, and its combination with the cavity method, see his article in this volume [12]. A topic I only shortly mention here is the dynamical behavior of spin glasses which is presented in much more depth by Guionnet [13].

## 2 The Basic Mathematical Models

The usual lattice spin–models of (nonrandom) Ising type are defined as follows. Consider a finite set $\Lambda$, and let $\Sigma_\Lambda \overset{\text{def}}{=} \{-1,1\}^\Lambda$. Let further $A = (a_{ij})_{i,j\in\Lambda}$ be a real symmetric matrix, and $\mathbf{h} = (h_i)_{i\in\Lambda}$ be a real vector. The Hamiltonian with these parameters is the mapping $H_{A,\mathbf{h}} : \Sigma_\Lambda \to \mathbb{R}$ defined by

$$-H_{A,\mathbf{h}}(\sigma) \overset{\text{def}}{=} \frac{1}{2}\sum_{i,j\in\Lambda} a_{ij}\sigma_i\sigma_j + \sum_{i\in\Lambda} h_i\sigma_i,$$

and the Gibbs measure $\mathcal{G}_{A,\mathbf{h}}$ on $\Sigma_\Lambda$ is defined by

$$\mathcal{G}_{\Lambda,A,\mathbf{h}}(\sigma) \overset{\text{def}}{=} \frac{1}{Z_{\Lambda,A,\mathbf{h}}} \exp\left[-H_{A,\mathbf{h}}(\sigma)\right], \tag{2.1}$$

where of course

$$Z_{\Lambda,A,\mathbf{h}} \overset{\text{def}}{=} \sum\nolimits_\sigma \exp\left[-H_{A,\mathbf{h}}(\sigma)\right] \tag{2.2}$$

is the so-called partition function, the normalizing factor in order that the Gibbs distribution is a probability distribution. (In the physics literature, one takes the Hamiltonian with a minus sign, so I keep with this tradition, although it is mathematically a bit annoying.) Of great importance is the (finite volume) free energy, defined by

$$F_\Lambda(A,\mathbf{h}) = \frac{1}{|\Lambda|}\log Z_{\Lambda,A,\mathbf{h}}. \tag{2.3}$$

The importance of this quantity is coming from the fact that most of the physical interesting quantities can be expressed through it, like mean magnetization, entropy, etc. For instance

$$\frac{\partial}{\partial h_j}F_\Lambda(A,\mathbf{h}) = \frac{1}{|\Lambda|}\sum\nolimits_\sigma \sigma_j \mathcal{G}_{\Lambda,A,\mathbf{h}}(\sigma),$$

and summing over $j \in \Lambda$ gives the mean magnetization under the Gibbs measure.

As for a finite set $\Lambda$, detailed properties of such models are usually impossible to describe, one usually tries to perform the "thermodynamic limit." For instance, if $\Lambda \subset \mathbb{Z}^d$, one can usually prove that the limiting free energy

$$f(A,\mathbf{h}) \stackrel{\text{def}}{=} \lim_{\Lambda\uparrow\mathbb{Z}^d} F_\Lambda(A,\mathbf{h})$$

exists, provided the $\Lambda$ approach $\mathbb{Z}^d$ in a not too nasty way, and $A$ and $\mathbf{h}$ are defined on the whole of $\mathbb{Z}^d$.

The best known example is the Ising model where $\Lambda$ is a finite (large) box in $\mathbb{Z}^d$, and

$$a_{ij} \stackrel{\text{def}}{=} \begin{cases} \beta & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

$\beta$ the so-called inverse temperature.

Short-range models are usually rather difficult to analyze, and often a qualitatively good approximation is obtained from *mean-field models* where every spin interacts with any other one on equal footing. The simplest mean-field model is the *Curie–Weiss model*. Here

$$a_{ij} \stackrel{\text{def}}{=} \beta/|\Lambda|, \ \forall i,j\in\Lambda.$$

In that case one has with $N \stackrel{\text{def}}{=} |\Lambda|$

$$\frac{1}{2}\sum\nolimits_{i,j\in\Lambda} a_{ij}\sigma_i\sigma_j = \frac{\beta}{2N}\left\{\sum\nolimits_{i\in\Lambda}\sigma_i\right\}^2,$$

and anything one wants to know can be derived from the Stirling approximation, and it becomes an easy exercise in elementary probability. If $k$ has the same parity as $N$, then

$$\#\left\{\sigma : \sum_{i\in\Lambda} \sigma_i = k\right\} = \frac{N!}{((N+k)/2)!\,((N-k)/2)!}$$
$$\simeq \left(\frac{N}{(N+k)/2}\right)^{(N+k)/2} \left(\frac{N}{(N-k)/2}\right)^{(N-k)/2}$$
$$= 2^N \exp\left[-N\left(\frac{1+\frac{k}{N}}{2}\log(1+k/N) + \frac{1-\frac{k}{N}}{2}\log\left(1-\frac{k}{N}\right)\right)\right].$$

From that one sees by a simple Laplace approximation that for constant $h_i = h$, one has that the limiting free energy of the Curie–Weiss model is given by

$$f(\beta, h) = \log 2 + \sup_{t\in[-1,1]}\left[\frac{\beta t^2}{2} + h - \frac{1+t}{2}\log(1+t) + \frac{1-t}{2}\log(1-t)\right].$$

In order to appreciate the simplification obtained by the mean-field ansatz, one has to compare that with the tremendously more difficult analysis in the ordinary Ising model as they can be found in standard textbooks, see, e.g., [14]. One aspect, one has however to keep in mind, is that for mean-field models it is difficult to talk about limiting Gibbs measures, "pure states," and the like. This aspect seems to have played a considerable role in the discussions and controversies whether mean-field spin glasses share some properties with short-range spin glasses. As I am not very knowledgeable on this subject, I do not want to comment about this issue, and rather advise the reader to read the contributions of Newman and Stein in this volume.

Spin glasses are models where the interactions are "disordered," which typically means that they are obtained as a random object. A topic which is still very poorly understood is the case of *short-range* random interactions, for instance when $\Lambda = \{-n, \ldots, n\}^d$, and the $a_{ij}$ are independent Gaussians for $|i-j| = 1$, and 0 otherwise. This is the *Edwards–Anderson model* on which there are ongoing controversial discussions in the physics community, the more so as it is very difficult to simulate it on computers with a reasonably large box and in interesting dimensions. One of the key issues is the presence of so-called "frustrations." This means that for three sites $i, j, k$, the interactions between $i$ and $j$ and between $j$ and $k$ may be positive, but between $i$ and $k$ negative. In particular, in contrast to the Ising model, spin glasses usually do not satisfy any of the well-known correlation inequalities, like the FKG inequality.

The situation is considerably better understood for the random field Ising model, where the interactions $a_{ij}$ are the same as for the Ising model, but where the $h_i$ are independent Gaussian random variables. On this, there are now classical results [15, 16], but we will not enter into this subject in this volume.

## 3 The Sherrington–Kirkpatrick Model

The Sherrington–Kirkpatrick model has the "mean-field" random Hamiltonian

$$-H_{N,\beta,h,\omega} \stackrel{\mathrm{def}}{=} \beta X_{n,\omega}(\sigma) + h\sum_{i=1}^{N}\sigma_i,$$

where

$$X_{n,\omega}(\sigma) \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{N}}\sum_{1\le i<j\le N} J_{ij}(\omega)\,\sigma_i\sigma_j,$$

and the $J_{ij}$ are independent standard Gaussian random variables, defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. (I will constantly use $\mathbb{P}$ for the probability measure governing the disorder, with $\mathbb{E}$ as the corresponding expectation.) We will often drop $\omega$ and $N$ in such expressions. One first observes that the $1/\sqrt{N}$-normalization is the right one in order to catch the "spirit" of a mean-field interaction: The total influence of the spins $\sigma_j$, $j\neq i$, on the $i$th spin is

$$\frac{1}{\sqrt{N}}\sum_{j>i} J_{ij}\sigma_j + \frac{1}{\sqrt{N}}\sum_{j<i} J_{ji}\sigma_j$$

which is of order 1.

Remark that for any $\sigma$, $X_N(\sigma)$ is a random variable, and indeed a centered Gaussian one. The covariances are given by

$$\begin{aligned}\mathbb{E}\left(H_N(\sigma)\,H_N(\sigma')\right) &= \frac{1}{N}\sum_{1\le i<j\le N}\sigma_i\sigma_j\sigma_i'\sigma_j' = \frac{1}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j\sigma_i'\sigma_j' - \frac{1}{2}\\ &= \frac{N}{2}\left(\frac{1}{N}\sum\nolimits_{i=1}^{N}\sigma_i\sigma_i'\right)^2 - \frac{1}{2}.\end{aligned} \tag{3.1}$$

The quantity in brackets is the so-called *overlap* of the two spin configurations

$$R_N(\sigma,\sigma') \stackrel{\mathrm{def}}{=} \frac{1}{N}\sum\nolimits_{i=1}^{N}\sigma_i\sigma_i'.$$

The (random) Gibbs distribution $\mathcal{G}_{N,\beta,h,\omega}$, the partition function $Z_{N,\beta,h,\omega}$, and the (finite $N$) free energy $F_{N,\beta,h,\omega}$ are defined as in (2.1)–(2.3).

There exist also variants where $h$ is a random variable or where $h\sum_{i=1}^{N}\sigma_i$ is replaced by $\sum_{i=1}^{N}h_i\sigma_i$, where the $h_i$ are random variables, e.g., $h_i = \gamma g_i + h$, $\gamma>0$, $h\in\mathbb{R}$, and the $g_i$ again being independent standard Gaussian random variables. This generalization is actually important, because the more general version appears naturally in the interpolation scheme invented by Guerra (see Sect. 4.1).

The free energy $F_N$ is still a random variable, and we write

$$f_N(\beta,h) \stackrel{\mathrm{def}}{=} \mathbb{E}F_{N,\beta,h},$$

the so-called "quenched" free energy. Sometimes, "quenched" refers to the random quantity only, but there is not much difference, as we will explain. In contrast, the so-called "annealed" free energy is obtained by taking the expectation inside the logarithm. By Jensens's inequality, $f_N$ is dominated by the annealed free energy.

The model is evidently closely connected with questions probabilists have been interested in for a long time, namely maxima (or minima) of (Gaussian) random vectors. For instance, $\lim_{\beta\to\infty} (1/\beta) \log Z_{N,\beta,0}$ is simply $\max_\sigma H_N(\sigma)$, which is just the maximum of a family of correlated Gaussians with a simple covariance structure. Probabilists have developed methods to investigate such questions for a long time, e.g., Dudley, Fernique, Talagrand, and many others. It is not difficult to see that $\max_\sigma H_N(\sigma)$ is of order $N$ and to prove that there are constants $0 < C_1 < C_2$ satisfying

$$\lim_{N\to\infty} \mathbb{P}\left(C_1 N \le \max_\sigma H_N(\sigma) \le C_2 N\right) = 1.$$

However, the standard probabilistic techniques cannot derive the exact constant, which the Parisi theory does, revealing a marvelous mathematical structure behind which is still *very* poorly understood, to this day.

### 3.1 Basic Properties of the SK-Model

The first question one typically answers is the existence of the free energy in the thermodynamical limit (here just $N \to \infty$). It is, however, not at all clear that the free energy

$$\lim_{N\to\infty} F_N(\beta, h)$$

exists. In principle, even if the limit exists, it could be a random variable. This possibility is, however, ruled out by Gaussian concentration inequalities. One says that the free energy is "self-averaging," meaning that no randomness remains in the $N \to \infty$ limit. For a proof of the following inequality, see for instance [17].

**Proposition 3.1.** *Let $\gamma_n$ be the standard Gaussian distribution on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipshitz continuous function with Lipshitz constant* 1. *Then for any $u > 0$*

$$\gamma_n\left(f > \int f d\gamma_n + u\right) \le \exp\left[-u^2/2\right].$$

If we apply this inequality to $F_N(\beta, h)$, regarded as a function of the standard Gaussian vector $(J_{ij})_{1\le i<j\le N}$, then one gets

$$\mathbb{P}\left(\left|\frac{1}{N}\log Z_{N,\beta,h} - \frac{1}{N}\mathbb{E}\log Z_{N,\beta,h}\right| \ge N^{-1/4}\right) \le 2\exp\left[-\frac{N^{1/2}}{\beta^2}\right].$$

It is therefore clear that instead of investigating $\lim_{N\to\infty} F_N(\beta, h)$, one can as well investigate the nonrandom object $\lim_{N\to\infty} f_N(\beta, h)$. The existence of this limit had been open for a long time, until Guerra and Toninelli [4] found a very nice, and not so obvious superadditivity property

$$\mathbb{E}\log Z_{N_1+N_2} \geq \mathbb{E}\log Z_{N_1} + \mathbb{E}\log Z_{N_2}, \tag{3.2}$$

from which one easily derives that

$$f(\beta, h) = \lim_{N\to\infty} f_N(\beta, h)$$

exists.

For the SK-model, the inequality came somewhat as a surprise. The proof is by a simple but very clever interpolation scheme which interpolates between the $(N_1 + N_2)$ system and the two independent smaller systems. Such interpolation schemes are at the very base of the recent progress in the understanding of the SK-model, as we will see later.

There are many quantities in the SK-model which are *not* self-averaging in the $N \to \infty$ limit, i.e., which stay random (or at least are believed to be so). An example is the overlap of two independent "replicas." Take $\sigma, \sigma'$ to be two independent realizations under $\mathcal{G}_{N,\beta,h,\omega}$ for a fixed $\omega$, and calculate $R_N(\sigma, \sigma')$, and then take the Gibbs expectation. This is still a random variable (being a function of the interaction strengths). For small $\beta$, these random variables have a nonrandom limit for $N \to \infty$, but the limit stays random for large $\beta$. The case $h = 0$ has some evident symmetry properties which make life easier, particularly in the high-temperature region.

For $h = 0$ and small enough $\beta$, the ("quenched") free energy equals the "annealed" free energy, a fact first proved in [18, 19].

**Theorem 3.2.** *For $h = 0$, and $\beta \leq 1$, one has*

$$f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}Z_{N,\beta} = \frac{\beta^2}{4} + \log 2. \tag{3.3}$$

The second equation is evident

$$\begin{aligned}\mathbb{E}Z_{N,\beta} &= \sum\nolimits_\sigma \mathbb{E}\exp\left[\beta H_N(\sigma)\right] = \sum\nolimits_\sigma \exp\left[\frac{\beta^2}{2}\operatorname{var}\left(H_N(\sigma)\right)\right] \\ &= 2^N \exp\left[\frac{\beta^2}{2}\operatorname{var}\left(H_N(\sigma)\right)\right] = 2^N \exp\left[\frac{\beta^2}{2}\left(\frac{N}{2} - \frac{1}{2}\right)\right]\end{aligned}$$

from which the claim follows. The somewhat astonishing fact is that one can interchange the expectation with the logarithm. Of course, by Jensen, one always has

$$\mathbb{E}\log Z_{N,\beta} \leq \log \mathbb{E}Z_{N,\beta}, \tag{3.4}$$

and therefore $f(\beta) \leq \beta^2/4 + \log 2$. We will indeed show later that $f(\beta) < \beta^2/4 + \log 2$ for $\beta > 1$. The proof of the above result is surprising simple and can be done by a second moment computation, proving that $\mathbb{E}Z^2 \leq \text{const} \times (\mathbf{E}Z)^2$ for $\beta < 1$. This second moment estimate is easy

$$\begin{aligned}\mathbb{E}Z^2 &= \sum_{\sigma,\tau} \mathbb{E}\exp\left[\frac{\beta}{\sqrt{N}}\sum\nolimits_{i<j} J_{ij}\left(\sigma_i\sigma_j+\tau_i\tau_j\right)\right] \\ &= \exp\left[\frac{\beta^2 N}{2}\right]\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{N}\sum\nolimits_{i<j}\sigma_i\sigma_j\tau_i\tau_j\right] \\ &= (\mathbb{E}Z)^2\left[2^{-N}\sum\nolimits_{\sigma}\exp\left[\frac{\beta^2 N}{2}\left(\frac{1}{N}\sum\nolimits_i \sigma_i\right)^2-\frac{\beta^2}{2}\right]\right],\end{aligned}$$

and the part in brackets is bounded for $\beta < 1$, by a simple Curie–Weiss coin tossing computation. Together with Gaussian isoperimetry (Proposition 3.1.), this proves (3.3). The original proofs in [18], and [19] were more complicated, but they derived also a much more detailed picture of the remaining fluctuations of $\log Z_N$.

There are other models like directed polymers for which one can prove that the quenched free energy equals the annealed one in certain regions, but typically, this is not possible by a simple second moment method in the full region where it is true. The fact that a second moment computation gives the result in the SK-model up to the correct critical value (for $h = 0$) is rather surprising. For $h \neq 0$, "quenched = annealed" is never true, which reveals that this is a much more interesting situation, even where $\beta$ is small.

## 3.2 The Replica Computation

The replica trick consists in the observation that for a positive number $x$, one has

$$\log x = \frac{d}{dn}\exp\left(n\log x\right)\Big|_{n=0} = \lim_{n\downarrow 0}\frac{x^n-1}{n}.$$

If $X$ is positive random variable, one therefore has, provided the interchange of limits is justified

$$\mathbb{E}\log X = \lim_{n\downarrow 0}\frac{\mathbb{E}X^n - 1}{n}.$$

As integer moments are often more easily evaluated then noninteger ones, the "trick" therefore is to evaluate $\mathbb{E}X^n$ for integer $n$, somehow extend things analytically, and performs the above limit.

For the SK-model, this is not quite the way, it is done. In fact, one just *starts* the computation of $\mathbb{E}Z_N^n$ assuming that $n$ is an integer, but as soon as it seems convenient, one gives up this illusion and lets $n \to 0$, *before* really finishing the computation. The calculation is easy, but it is hard to swallow for a mathematician. Here it is

$$\mathbb{E}Z^n = 2^{nN}\mathbb{E}\,\mathrm{tr}_\sigma\exp\left[\frac{\beta}{\sqrt{N}}\sum\nolimits_{1\le i<j\le N} J_{ij}\sum\nolimits_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha + h\sum\nolimits_{\alpha=1}^{n}\sum\nolimits_{i=1}^{N}\sigma_i^\alpha\right],$$