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Karel Dekimpe

Almost-Bieberbach Groups: Affine and Polynomial Structures

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For Katleen, Charlotte and Sofie.

Preface

The reader taking a first glance at this monograph might have the (wrong) impression that a lot of topology/geometry is involved. Indeed, the objects we study in this book are a special kind of manifold, called the infra–nilmanifolds. This is a class of manifolds that can, and should, be viewed as a generalization of the flat Riemannian manifolds. However, the reader familiar with the theory of the flat Riemannian manifolds knows that such a manifold is completely determined by its fundamental group. Moreover, the groups that occur as such a fundamental group can be characterized in a purely algebraic way. More precisely, a group E is the fundamental group of a flat Riemannian manifold if and only if E is a finitely generated torsion free group containing a normal abelian subgroup of finite index. These groups are called Bieberbach groups. It follows that one can study the flat Riemannian manifolds in a purely algebraic way.

This group theoretical approach is also possible for the infra–nilmanifolds, which are obtained as a quotient space under the action of a group E on a simply connected nilpotent Lie group G, where E acts properly discontinuously and via isometries on G. (If G is abelian, then this quotient space is exactly a flat Riemannian manifold). The fundamental group of an infra–nilmanifold is referred to as an almost–Bieberbach group. It turns out that much of the theory of Bieberbach groups extends to the almost–Bieberbach groups. Thus for instance, a group E is the fundamental group of an infra–nilmanifold if and only if E is a finitely generated torsion free group containing a normal nilpotent subgroup of finite index.

The aim of this book is twofold:

1. I wish to explain and describe (in full detail) some of the most important group–theoretical properties of almost–Bieberbach groups.

I have the impression that the algebraic nature of almost–Bieberbach groups is far from well known, although many of their properties are just a straightforward generalization of the corresponding properties of the Bieberbach groups. On the other hand, I do not claim to be a specialist of Bieberbach (or more general crystallographic) groups and so a lot more of the theory of Bieberbach (crystallographic) groups still has to be generalized. I hope therefore that this book might stimulate the reader to help in this generalization.

2. I also felt there is a need for a detailled classification of all almost–Bieberbach groups in dimensions ≤ 4. We will see that an infra–nilmanifold is completely determined by its fundamental group. So my classification of almost–Bieberbach groups can also be viewed as a classification of all infra–nilmanifolds of dimensions ≤ 4. I myself use the tables of almost–Bieberbach groups not really as a classification but as an elaborated set of examples or "test cases" for new hypotheses. I hope that, one day, they can be of the same value to you too.

I tried to write this monograph both for topologists/geometers as for algebraists. Therefore, I made an effort to keep the prerequisites as low as possible. However, the reader should have at least an idea of what a Lie group is. Also, a little knowledge of the theory of covering spaces can be helpful now and then. From the algebraic point of view, I assume that the reader is fairly familiar with nilpotent groups and that he is acquainted with group extensions and its relation to cohomology of groups.

Although this work is divided into eight chapters, there are really three parts to distinguish.

1. In the first part (Chapter 1 to Chapter 3), we define almost–crystallographic and almost–Bieberbach groups. We spend a lot of time in providing alternative definitions for them. Also we show how the three famous theorems of L. Bieberbach on crystallographic groups can be generalized to the case of almost–crystallographic groups. These first chapters could already suffice to let the reader start his own investigation of almost–crystallographic groups.

2. Chapter 4 forms a part on its own. It deals mainly with my own field of interest, namely the canonical type representations. These are representations of a polycyclic–by–finite group (in our situation always virtually nilpotent), which respect in some sense a given

filtration of that group. We discuss both affine and polynomial representations and present some nice existence and uniqueness results. The reason for considering polycyclic–by–finite groups is natural in the light of Auslander's conjecture.

3. The last part of this monograph (Chapter 5 to Chapter 8) describes a way to classify almost–Bieberbach groups. We also give a complete list of all almost–Bieberbach groups in dimensions ≤ 4, which were obtained using the given method. Moreover, we show how it was possible to use these tables and find in a pure algebraic way some topological invariants (e.g. Betti numbers) of the corresponding infra–nilmanifolds.

Finally, I would like to say a few words of thanks. To Professor Paul Igodt who introduced me to the world of infra–nilmanifolds and who proposed me to investigate the possibility of classifying the almost–Bieberbach groups. I am also grateful to Professor Kyung Bai Lee, since I owe much of my knowledge on almost–Bieberbach groups to him. But most of all I must thank my wife Katleen, for her encouragement when I was doing mathematics in general and especially for her support and practical help when I was writing this book.

Karel Dekimpe,
Kortrijk, August 19, 1996

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Chapter 1

Preliminaries and notational conventions

1.1 Nilpotent groups

In this first chapter we discuss the fundamental results needed to understand this book. Our primary objects of study are virtually nilpotent groups. Remember that a group G is said to be virtually $\mathcal{P}$, where $\mathcal{P}$ is a property of groups, if and only if G contains a normal subgroup of finite index which is $\mathcal{P}$.

Although we assume familiarity with the concept of a nilpotent group, we recall some special aspects of this theory in order to fix some notations.

Let N be any group, then the upper central series of N

$$Z_*(N): \ Z_0(N) = 1 \subseteq Z_1(N) \subseteq \cdots \subseteq Z_i(N) \cdots$$

is defined inductively by the condition that

$$Z_{i+1}(N)/Z_i(N) = Z(N/Z_i(N))$$

where $Z(G)$ denotes the center of a group G. The group N is said to be nilpotent if the upper central series reaches N after a finite number of steps, i.e. there exists a positive integer c such that $Z_c(N) = N$. If c is the smallest positive integer such that $Z_c(N) = N$, we say that N is c-step nilpotent or N is nilpotent of class c.

Another frequently used central series is the lower central series. This series uses the commutator subgroups of a group N. We use the convention that the commutator $[a,b] = a^{-1}b^{-1}ab$ for all $a, b \in N$. Conjugation in N with a is indicated by $\mu(a)$. Sometimes we use $b^a = a^{-1}ba = \mu(a^{-1})(b)$.

The lower central series of N is the central series

$$N = \gamma_1(N) \supseteq \gamma_2(N) \supseteq \cdots \supseteq \gamma_i(N) \supseteq \cdots$$

where the i–fold commutator subgroups $\gamma_i(N)$ are defined inductively by the formula

$$\gamma_{i+1}(N) = [N, \gamma_i(N)].$$

The groups we are interested in are the finitely generated torsion free nilpotent groups for which it makes sence to consider central series

$$N_* : \ 1 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_c = N$$

with torsion free quotients N_i/N_{i-1} for $1 \leq i \leq c$. We will refer to such a central series N_* as a torsion free central series.

Given such a torsion free central series, there exists integers $k_i \in \mathbb{Z}$ such that $N_i/N_{i-1} \cong \mathbb{Z}^{k_i}$. We write $K_i = \sum\limits_{j \geq i} k_j$. We also write K for K_1, which is the rank or Hirsch number of N.

A set of generators

$$\{a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, a_{3,1}, \ldots, a_{c,k_c}\}$$

of N will be called compatible with N_* iff

$$\forall i \in \{1, 2, \ldots, c\} : \ a_{1,1}, a_{1,2}, \ldots, a_{i,k_i} \text{ generate } N_i.$$

It is clear at once that any torsion free central series of N admits a compatible set of generators. Such a compatible set of generators may be obtained in the following way: First we choose k_1 generators of N_1, say $a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}$. Then we complete this set to a set of generators for N_2. So we have to choose elements $a_{2,1}, \ldots a_{2,k_2}$. We continue this way and finally we find the last k_c generators $a_{c,1}, \ldots, a_{c,k_c}$. Any element $n \in N$ can now be written uniquely in the form

$$n = a_{c,1}^{x_{c,1}} a_{c,2}^{x_{c,2}} \ldots a_{1,k_1}^{x_{1,k_1}} \in N, \text{ for some } x_{i,j} \in \mathbb{Z}.$$

This shows that we may identify n with its coordinate vector

$$(x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{i,j}, \ldots, x_{c,k_c}) \in \mathbb{Z}^K.$$

For all torsion free finitely generated nilpotent groups N, the upper central series determines a torsion free central series, while in general the lower central series fails to have torsion free factors. However, we can alter the lower central series slightly in order to get a torsion free central series. To explain this we need the concept of the isolator:

Definition 1.1.1 *(see also [56], [59])*
Let G be a group. For H a subgroup of G, the **isolator** *of H in G (sometimes called the* **root set***) is defined by*

$$\sqrt[G]{H} = \{g \in G \parallel g^k \in H \text{ for some } k \geq 1\}.$$

In general, the isolator of a subgroup H in G doesn't have to be a subgroup itself. E.g. if $H = 1$ then $\sqrt[G]{H}$ is exactly the set of torsion elements of G, which needn't be a group in general. We will only need the isolator of a commutator subgroup.

Lemma 1.1.2 *Let G be any group. Then,*

1. *$\forall k \in \mathbb{N}_0$: $\sqrt[G]{\gamma_k(G)}$ is a characteristic subgroup of G.*

2. *$\forall k \in \mathbb{N}_0$: $G/\sqrt[G]{\gamma_k(G)}$ is torsion free.*

3. *$\forall k, l \in \mathbb{N}_0$: $[\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \subseteq \sqrt[G]{\gamma_{k+l}(G)}$.*

For the proof of this lemma we refer the reader to [56, page 473].

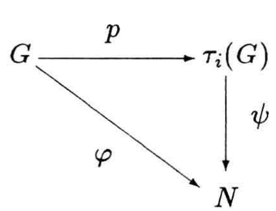
It follows that for any finitely generated, torsion free c-step nilpotent group N the series

$$\sqrt[N]{\gamma_{c+1}(N)} = 1 \subseteq \sqrt[N]{\gamma_c(N)} \subseteq \cdots \subseteq \sqrt[N]{\gamma_2(N)} \subseteq \sqrt[N]{\gamma_1(N)} = \sqrt[N]{N} = N$$

is a torsion free central series. We will refer to this series as **the adapted lower central series.**

For any group G, the groups $\sqrt[G]{\gamma_i(G)}$ can be determined by means of a universal property. Write $\tau_i(G) = G/\sqrt[G]{\gamma_{i+1}(G)}$ and denote the canonical projection of G onto $\tau_i(G)$ by p. The group $\tau_i(G)$ is the biggest possible torsion free quotient of G, which is nilpotent of class $\leq i$. Formally

Lemma 1.1.3 Universal property of $\tau_i(G)$.
Let G be any group and suppose that N is a torsion free nilpotent group of class $\leq i$. Given a group homomorphism $\varphi : G \to N$, there exists a unique morphism $\psi : \tau_i(G) \to N$ such that $\varphi = \psi \circ p$. I.e. the following diagram commutes:



Proof: As N is nilpotent of class $\leq i$, all $i+1$–fold commutators are mapped trivially. So the morphism φ factors through $G/\gamma_{i+1}(G)$. Also, as N is torsion free, the characteristic subgroup $\tau(G/\gamma_{i+1}(G))$ consisting of all torsion elements of $G/\gamma_i(G)$ is mapped trivially. Therefore, there is a factorization

$$\varphi : G \to G/\gamma_{i+1}(G) \to (G/\gamma_{i+1}(G))/\tau(G/\gamma_{i+1}(G)) \to N. \qquad (1.1)$$

But $\sqrt[G]{\gamma_{i+1}(G)}$ consists exactly of those elements which are mapped into the set of torsion elements $\tau(G/\gamma_{i+1}(G))$ under the canonical projection of G onto $G/\gamma_{i+1}(G)$. So,

$$\tau(G/\gamma_{i+1}(G)) = \sqrt[G]{\gamma_{i+1}(G)}/\gamma_{i+1}(G)$$

from which it follows that the factorization (1.1) mentioned above can be written as:

$$\varphi : G \xrightarrow{p} \tau_i(G) \xrightarrow{\psi} N.$$

This establishes the existence of the map ψ. The uniqueness is obvious.

■

The above proposition determines the subgroup $\sqrt[G]{\gamma_{i+1}(G)}$ of G completely. For suppose there exists another normal subgroup A of G (together with a canonical projection $q : G \to G/A$), such that any morphism $\varphi : G \to N$ as above can be written in the form $\varphi = \psi' \circ q$. In this case let N be equal to $G/\sqrt[G]{\gamma_{i+1}(G)}$ and let $\varphi = p$. It is obvious that the map ψ' in this case maps the coset $g\,A$ onto $g\,\sqrt[G]{\gamma_{i+1}(G)}$, for all $g \in G$. By reversing the roles, we obtain a morphism $\psi : G/\sqrt[G]{\gamma_{i+1}(G)} \to A : g\,\sqrt[G]{\gamma_{i+1}(G)} \to g\,A$, which is the inverse of ψ'. Therefore, the groups A and $\sqrt[G]{\gamma_{i+1}(G)}$ coincide. As an application of the above universal property we find:

Lemma 1.1.4 *Let G be any group.*
For all $j \geq i$, there is a canonical isomorphism

$$\tau_i(\tau_j(G)) \cong \tau_i(G).$$

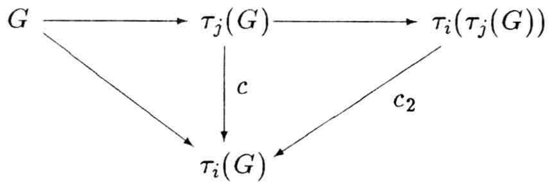
It follows that for $\overline{G} = \sqrt[G]{\gamma_{j+1}(G)}$

$$\sqrt[G/\bar{G}]{\gamma_{i+1}(G/\bar{G})} = \sqrt[G]{\gamma_{i+1}(G)}/\bar{G}.$$

Proof: By the universal property of $\tau_i(G)$ there is a canonical morphism

$$c_1 : \tau_i(G) \to \tau_i(\tau_j(G))$$

which maps the coset of an element g of G onto the coset of $g\,\overline{G}$ in $\tau_i(\tau_j(G))$. Conversely, we have the following commutative diagram



where

1. the non labeled arrows are canonical projections onto a quotient group.
2. c is induced by the fact that $\sqrt[G]{\gamma_{j+1}(G)}$ is contained in $\sqrt[G]{\gamma_{i+1}(G)}$ $(j \geq i)$.
3. c_2 is obtained by the universal property of $\tau_i(\tau_j(G))$.

It is clear now that c_1 and c_2 are each others inverse. The last claim of the lemma, concerning the equality of the two subgroups of $G/\overline{G}$, follows from the comments preceding this lemma.

■

The following technical lemmas will be needed at special occasions during our treatment of almost–crystallographic groups.

Lemma 1.1.5 *Let H be a torsion free, normal subgroup of finite index in a group G. Assume $z \in Z(H)$ and $x \in G$ such that $[x, z] \neq 1$. Then any commutator of the form $[x, [\ldots[x, [x, z]]\ldots]$ is not trivial.*

<u>Proof :</u> Consider the sequence $(c_i)_{i \in \mathbb{N}}$ in $Z(H)$ defined by $c_0 = z$ and $c_{i+1} = [x, c_i]$. We proceed by induction. Assume $c_i \neq 1$ and $c_{i+1} = 1$ for $i \geq 1$. If $[G : H] = m$, $x^m \in H$ and hence it commutes with c_{i-1}. A trivial computation shows that

$$1 = [x^m, c_{i-1}] = \prod_{j=1}^{m} [x, c_{i-1}]^{x^{m-j}} = \prod_{j=1}^{m} c_i^{x^{m-j}} = c_i^m.$$

Since H is torsion free $c_i = 1$, which is a contradiction.

■

Lemma 1.1.6 *If $0 \to H \to G \to K \to 1$ determines G as a central extension of an abelian group H by a group K which is nilpotent of class $\leq c$, then G is nilpotent of class $\leq (c + 1)$.*

The proof is straightforward and left to the reader.

Lemma 1.1.7 *If $1 \to H \to G \to \mathbb{Z}_k \to 1$ determines G as a nilpotent extension of a torsion–free, c-step nilpotent group H by a finite cyclic group, then G itself is c-step nilpotent.*

Proof : The proof goes by induction on the nilpotency class of H. So, assume H is abelian. Take $g \in G$ and $h \in H$. If $[g,h] \neq 1$, lemma 1.1.5 implies that G is not nilpotent. Consequently, it follows that the extension is a central one. Having chosen a section $s : \mathbb{Z}_k \to G$, elements $g \in G$ can be written as $hs(x^l)$ ($h \in H$, x is a generator of $\mathbb{Z}_k$ and $0 \leq l \leq k-1$). Now, it is clear that if g_1 and g_2 are in G, $[g_1, g_2] = [s(x^{l_1}), s(x^{l_2})]$. Since $s(x)^l$ and $s(x^l)$ belong to the same coset of H, it follows that $[g_1, g_2] = [s(x)^{l_1}, s(x)^{l_2}] = 1$, so G is abelian.

Now assume H is of class c. Lemma 1.1.5 implies that $Z(H) \subset Z(G)$ when G is nilpotent. Consider the short exact sequence $1 \to H/Z(H) \to G/Z(H) \to \mathbb{Z}_k \to 1$. Here, $H/Z(H)$ is torsion–free $(c-1)$-step nilpotent. By induction $G/Z(H)$ itself is $(c-1)$-step nilpotent. Apply lemma 1.1.6 to the extension $1 \to Z(H) \to G \to G/Z(H) \to 1$ and deduce that G is nilpotent of class $\leq c$.

■

Lemma 1.1.8 *Let G be any group and suppose that T is a torsion free normal subgroup of G, while F is a finite normal subgroup of G. Then*

$$[T, F] = 1.$$

Proof: Let $t \in T$ and $f \in F$, then

$$[t, f] = \underbrace{t^{-1} f^{-1} t}_{\in F} f = t^{-1} \underbrace{f^{-1} t f}_{\in T} \in T \cap F = 1$$

■

Lemma 1.1.9 *Let φ be any automorphism of $\mathbb{Z}^k$. If there exists a subgroup A of finite index in $\mathbb{Z}^k$ on which φ is the identity, then φ is the identity automorphism.*

Proof: φ can be represented by an invertible matrix M with integral entries. Seen as an element of $\mathrm{Gl}(n, \mathbb{R})$, this matrix represents a linear mapping leaving fixed a generating set (i.e. A) of the real vector space $\mathbb{R}^n$. This implies that M is the $n \times n$–identity matrix.

■

1.2 Nilpotent Lie groups

Although we intend to keep the topological/geometrical background needed to understand this book as small as possible, we need at least some knowledge concerning nilpotent Lie groups. In fact, most of the stuff we will use can be found in the magnificent paper of A.I. Mal'cev [51]. We refer to this paper for all the proofs of the claims we make here.

Throughout this section G will denote a connected and simply connected nilpotent Lie group. We use $\mathfrak{g}$ to indicate the Lie algebra of G. This Lie algebra $\mathfrak{g}$ has the same dimension and nilpotency class as G. Moreover, in the case of connected and simply connected nilpotent Lie groups it is known that the exponential map $\exp : \mathfrak{g} \to G$ is bijective. We denote its inverse by log. The exponential map earns its name because of the fact that for matrix groups/algebras the exponential map is indeed given by the exponentiation of matrices. I.e. $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$.
If H is another connected and simply connected nilpotent Lie group, with Lie algebra $\mathfrak{h}$, then we have the following properties:

- For any morphism $\varphi : G \to H$ of Lie groups, there exists a unique morphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$ (differential of φ) of Lie algebras, making the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \log \downarrow \uparrow \exp & & \log \downarrow \uparrow \exp \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \end{array} \qquad (1.2)$$

- Conversely, for any morphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras, there exists a unique morphism $\varphi : G \to H$ of Lie groups, making the above diagram commutative.

In G, it makes sense to speak of a^x where $a \in G$ and $x \in \mathbb{R}$. (E.g. consider the one-parameter subgroup of G passing through a). A formal definition may look as follows:

Definition 1.2.1

$$a^x = \exp(x \log a), \quad \forall a \in G, \ \forall x \in \mathbb{R}.$$

The definition satisfies all the expected conditions:

1. $a.a^{-1} = a^{-1}.a = 1; \ \ a^0 = 1,$

2. $a^n = \underbrace{a.a\ldots a}_{n \text{ times}}$, if $n \in \mathbb{N}$,

3. $\left(a^{\frac{1}{n}}\right)^n = a$, if $n \in \mathbb{N}$,

4. $(a^x)^y = a^{xy}$,

5. $a^x.a^y = a^{x+y}$,

6. If $\varphi : G \to H$ is a morphism between connected and simply connected nilpotent Lie groups, then $\varphi(a^x) = (\varphi(a))^x$.

We give a proof of this last property, using the commutative diagram (1.2):

$$\begin{aligned} \varphi(a^x) &= \varphi(\exp x \log a) \\ &= \exp(d\varphi(x \log a)) \\ &= \exp(x d\varphi(\log a)) \\ &= \exp(x \log(\varphi(a))) \\ &= (\varphi(a))^x . \end{aligned}$$

■

We also mention the famous Campbell–Baker–Hausdorff formula:

$$\forall A, B \in \mathfrak{g} : \; \exp(A). \exp(B) = \exp(A * B), \tag{1.3}$$

where

$$A * B = A + B + \frac{1}{2}[A, B] + \sum_{m=3}^{\infty} C_m(A, B).$$

Here $C_m(A, B)$ stands for a rational linear combination of m–fold Lie brackets in A and B. Since our Lie algebras are nilpotent, the sum involved in $A * B$ is always finite. As an immediate consequence of this formula, one sees that

$$\forall a, b \in G : \; log(a.b) = \log a * \log b.$$

Of major importance to us, is the concept of a uniform lattice of G.

Definition 1.2.2 *Let G be a connected and simply connected nilpotent Lie group. A uniform lattice of G is a uniform discrete subgroup, i.e. a discrete subgroup with compact quotient, N of G.*

We remark that not all connected and simply connected nilpotent groups admit lattices.
One of the nicests results of Mal'cev is the "unique isomorphism extension property"

Theorem 1.2.3 *Let G and H be two connected and simply connected nilpotent Lie groups. Suppose moreover that N and M are uniform lattices of G and H respectively. Then any isomorphism $\varphi : N \to M$ extends uniquely to an isomorphism of Lie groups of G onto H.*

In case we use this property for $M = N$, we also say "the unique automorphism extension property ".

V.V. Gorbacevič ([33]) generalized this theorem as follows:

> Let N be a uniform lattice of a simply connected, connected nilpotent Lie group G and let H be an arbitrary simply connected, connected nilpotent Lie group. Then any morphism $\varphi : N \to H$ extends uniquely to a morphism $G \to H$.

Mal'cev also describes all possibilities of uniform lattices in a connected and simply connected nilpotent Lie group.

Theorem 1.2.4 *Any lattice N of a connected and simply connected nilpotent Lie group G is a finitely generated torsion free nilpotent group. Conversely, for any torsion free finitely generated nilpotent group N there exists (up to isomorphism) exactly one connected and simply connected nilpotent Lie group G containing N as a uniform lattice. We refer to this G as the* **Mal'cev completion** *of N.*

Let N be a torsion free and finitely generated nilpotent group with a torsion free central series N_* as in the previous section. Suppose moreover that

$$\{a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{c,k_c}\}$$

is a set of generators compatible with N_*. Then the elements

$$\{A_{1,1} = \log(a_{1,1}), A_{1,2} = \log(a_{1,2}), \ldots, A_{c,k_c} = \log(a_{c,k_c})\}$$

form a basis for the Lie algebra $\mathfrak{g}$ of the Mal'cev completion G of N. It follows that any element g of G can be written uniquely in the form

$$g = a_{c,1}^{x_{c,1}} a_{c,2}^{x_{c,2}} \ldots a_{1,k_1}^{x_{1,k_1}} \text{ for some } x_{i,j} \in \mathbb{R}.$$

Any element can thus be identified with a coordinate vector

$$(x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{i,j}, \ldots, x_{c,k_c}) \in \mathbb{R}^K.$$