Solving Problems in

VIBRATIONS

JS Anderson
M Bratos—Anderson

Solving Problems in Vibrations

J. S. Anderson B.Sc.(Eng), Ph.D., A.K.C., C.Eng., M.R.Ae.S.
Department of Mechanical Engineering,
The City University, London

M. Bratos-Anderson M.Sc., Dr.Tech.Sc.
Department of Mechanical Engineering,
The City University, London

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Preface

This book has been written to aid students during the second year of study for an engineering degree. The text is mostly concerned with linear systems with a single degree of freedom. A chapter on systems with two degrees of freedom has been included, even though this subject may be studied in the third year of some courses.

Each chapter of the book possesses three distinct parts. Firstly there is some introductory theory giving background to the worked examples which form the second and most important part. Finally, at the end of each chapter, there are exercises to enable the student to obtain experience in solving problems. The worked examples described in this book are mostly based on examination questions. We do not intend to provide model solutions of the kind that the student might hope to provide for his examiners. We aim to show how the principles of mechanics are applied to solving problems in vibration. In many of the worked examples we give a full explanation of each step in the solution procedure. We sometimes provide alternative solutions or digress to discuss interesting aspects of the problem.

In this book the method of solution is mostly through free-body diagrams which are used as an aid to writing the equations of motion. However, in Chapter 4 energy methods are also discussed. In writing the differential equations the 'dot' notation is used to indicate differentiation with respect to time. A bold symbol indicates a vector.

We are indebted to the Registrar of the City University for permission to include examination problems which have previously occurred in examination papers of the University. Many of these examination problems have been written by our colleagues in the Department of Mechanical Engineering at the City University. We have had a wide selection to choose from and we are very grateful to our colleagues for the work they have done over many years. In particular we should like to acknowledge the help of J. E. Cannell, G. T. S. Done, I. W. Graham, H. R. Harrison, B. G. Main, T. Nettleton, D. W. Oates, A. R. D. Thorley, R. G. P. Weighton and L. J. Wilkins.

Finally, we submit this book to our readers with a certain amount of humility. We realize that it is easy for questions to seem contrived and even ridiculous. A cautionary example is quoted by Max Born in his book *My Life: Recollections of a Nobel Laureate*, Scribner's 1978: 'On an elastic bridge stands an elephant of negligible mass; on his trunk sits a mosquito of mass m. Calculate the vibrations of the bridge when the elephant moves the mosquito by rotating his trunk.'

Contents

1

Free, undamped vibrations of systems with a single degree of freedom

1.1 Single degree of freedom system

A mechanical system that requires n co-ordinates to define its position at any instant in time is said to have n degrees of freedom. In Figure 1.1(a) the rigid body on the end of the spring can move up or down along the axis YY. This forms a rectilinear system in which one co-ordinate y is required to define the position of the body. The position of the body at any instant is hence defined in terms of the co-ordinate y, and the system is thus said to have one degree of freedom.

In Figure 1.1(b) a rotational system is shown with one degree of freedom. Here a light, elastic shaft, rigidly fixed at one of its ends, supports a rigid flywheel which has a certain moment of inertia. As the shaft twists, a line OA on the flywheel moves to OA′ through an angle θ. θ is the one co-ordinate which is sufficient to define the position of every point on the flywheel at any instant in time.

A compound pendulum is also a system with a single degree of freedom. The position of the centre of mass G can be defined if one

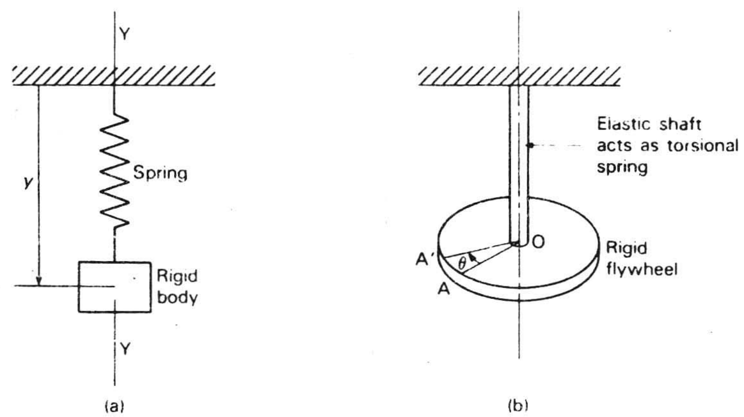

Figure 1.1 System with a single degree of freedom: (a) rectilinear, (b) torsional.

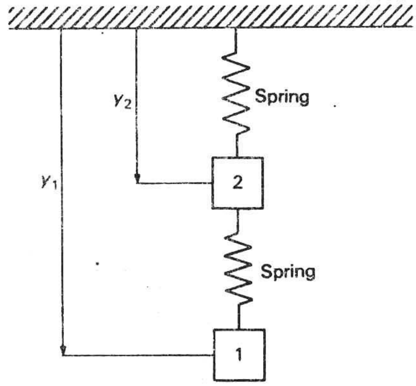

Figure 1.2 System with two degrees of freedom.

co-ordinate is known, namely the angle θ which is the angular displacement of the rigid body (compound pendulum) from its vertical position. See ahead to Figure 1.17 for a diagram of a compound pendulum.

However, the rectilinear system of Figure 1.2 has two degrees of freedom. If the rigid body 1 is displaced downwards or upwards there are still infinitely many possibilities for the position of body 2. Each of the two bodies may move independently in the vertical direction. Two co-ordinates y_1 and y_2 are needed to define the position of the system at any instant of time.

1.2 Undamped system

In this chapter we shall not take into account damping forces which cause energy dissipation in mechanical systems. There will be no loss of energy and once a vibration is started it will be maintained for all time. An undamped system with a single degree of freedom gives rise to a type of motion known as simple harmonic motion, in which the acceleration of the body is proportional to the displacement (y or θ) with negative sign.

1.3 Springs

A massless, elastic element is a fundamental component in the vibrating systems described in this book. In Figure 1.1(a) there is a diagrammatical representation of a helical spring with metal coils. Such a spring is normally linear, i.e., its extension or compression is proportional to the force applied. The ratio of applied force to resulting deflection is a constant called the spring constant or stiffness. Most rubber springs are non-linear and become stiffer (or harder) as a sufficiently large force is applied. Softening springs are also possible. The relationships between force and deflection for all cases are shown in Figure 1.3.

A rod or wire of elastic material acts as a linear spring, provided that the elastic limit is not exceeded. Let us assume that when a force F is applied along the axis of the rod or wire there is an extension or

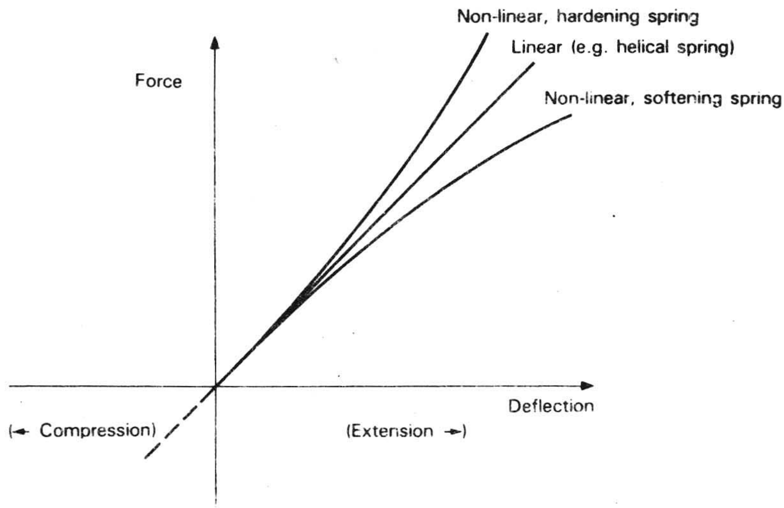

Figure 1.3 Force–deflection relationships for different springs.

compression y. The ratio F/y is the stiffness of the rod and is related to cross-sectional area A of the rod, its length l and Young's modulus E, a characteristic factor of the elastic material, in the following way:

$$\text{stiffness} = F/y = EA/l \qquad (\text{units} - \text{N/m}),$$

where

$$E = \text{stress/strain} = (F/A)/(y/l) = Fl/yA.$$

Similarly, a torsional spring in a rotational system (Figure 1.1(b)) has a rotational stiffness which is provided by the shaft (or wire) as it twists about its axis. This torsional stiffness is defined as the ratio of a torque T to an angle of twist θ induced in the shaft by the torque, and is given by:

$$T/\theta = GI_p/l \qquad (\text{units} - \text{N m/rad}),$$

where G is the modulus of rigidity, I_p is the polar second moment of area ($\pi d^4/32$ for a wire or circular solid shaft of diameter d) and l is the length of the wire or shaft which is twisting.

Example 1.1
Stiffness of beam

> A wooden beam of length 2 m is simply supported at its ends. The beam is rectangular in cross-section, 300 mm wide by 30 mm thick. Young's modulus E is 11×10^9 Pa. If a concentrated load of 300 kg is attached to the beam at the middle what is the equivalent stiffness of the beam?

Solution A standard textbook on strength of materials (for example, *Mechanics of Solids and Structures* by P. P. Benham and F. V. Warnock (2nd edn), Pitmans 1973, or *Strength of Materials and Structures* by

J. Case and A. H. Chilver, Edward Arnold 1971) will state that the deflection y at the centre of a simply supported beam of length l caused by a load F at the centre is given by:

$$y = Fl^3/(48EI),$$

where I is the second moment of area of the beam cross-section.

The equivalent stiffness $k = 48EI/l^3$. In this example,

$$\begin{aligned} I &= (300 \times 30)(30^2/12) \times 10^{-12}\ \text{m}^4 \\ &= 675 \times 10^{-9}\ \text{m}^4, \\ k &= (48 \times 11 \times 10^9 \times 675 \times 10^{-9})/2^3 \\ &= 44.55\ \text{kN/m}. \end{aligned}$$

Answer Stiffness of the beam is **44.55 kN/m.**

1.4 The basic differential equation of the motion

The spring mass system of Figure 1.1(a) is a basic arrangement. In the static equilibrium position the spring of stiffness k is extended by y_0 and the forces acting on the body are shown in Figure 1.4(a).

For the case of static equilibrium (see the free body diagram of Figure 1.4(a)):

$$mg - ky_0 = 0.$$

If the body is displaced by, for example, y, downwards from the equilibrium position, the forces are now as shown in Figure 1.4(b). There is no longer static equilibrium and the body accelerates. Applying Newton's second law, we obtain the following differential equation of motion of the system:

$$mg - k(y_0 + y) = m\ddot{y}$$

and finally

$$-ky = m\ddot{y}, \qquad [1.1]$$

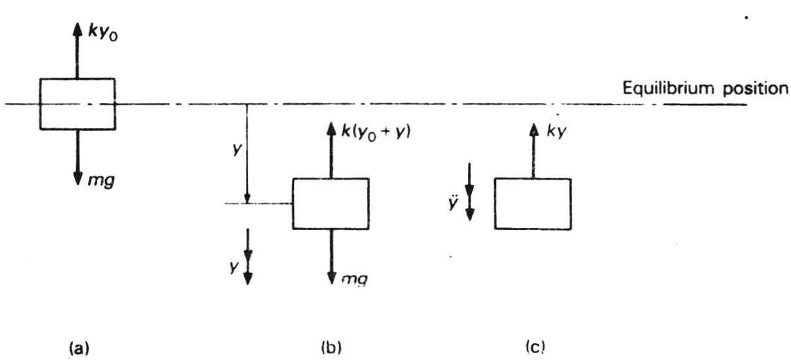

Figure 1.4 Free-body diagrams for body on spring: (a) in static equilibrium; (b) dynamic case showing all forces; (c) dynamic case showing only oscillating force.

which is the equation for simple harmonic motion and can be written as:

$$\ddot{y} + (k/m)y = 0. \qquad [1.2]$$

Equation (1.2) is a linear, second-order differential equation with constant coefficients.

For a given system the ratio k/m may be denoted by ω_n^2 so that we have:

$$\ddot{y} + \omega_n^2 y = 0. \qquad [1.3]$$

ω_n is called the undamped circular natural frequency and has the units of rad/s. It should not be confused with f_n, the undamped natural frequency, which has units of Hz (an abbreviation for hertz and previously known as cycles/second). With or without subscripts ω and f are related by the following expression:

$$\omega = 2\pi f.$$

You may find other expressions for circular frequency, such as angular frequency, radiancy or pulsatance.

It is possible to write down equation [1.1] immediately by analysing the free body diagram in Figure 1.4(c). The forces mg and ky_0 exist all the time, and always cancel each other out. Equation [1.1] only contains terms whose values oscillate during the vibration. In general solutions will be obtained more quickly by omitting these constant forces. However, there are some problems where the weight needs to be considered: for example, in problems involving inverted or compound pendula. In these cases the moment of the weight is varying.

1.5 Solution of the differential equation

The general solution of equation [1.3] is known to be in terms of trigonometric functions and two arbitrary constants. Thus,

$$y = A \cos \omega_n t + B \sin \omega_n t. \qquad [1.4]$$

An alternative formula is:

$$y = X \cos (\omega_n t - \Phi), \qquad [1.5]$$

where the two constants are now the amplitude X $(=(A^2 + B^2)^{0.5})$ and the phase angle Φ $(=\tan^{-1} B/A)$.

The values of the constants must be found from the initial conditions. Thus at $t = 0$ we need to know the values of y and $\dot{y}$, as two separate initial conditions are required for a second-order differential equation.

Example 1.2
Mass–spring system with different initial conditions

A body of mass 4 kg is supported on springs which have an equivalent stiffness of 2500 N/m. (a) The mass is initially displaced downwards by 100 mm from the equilibrium position and then released. (b) The body is struck by an impulse of 10 N s which acts vertically downwards. In both cases determine the expressions for the displacements.

Solution The diagrams of Figure 1.4 apply, so they will not be repeated. $m = 4$ kg and $k = 2500$ N/m, hence $\omega_n^2 = 625$ and $\omega_n = 25$ rad/s. Equation [1.3] becomes now:

$$\ddot{y} + 625y = 0$$

and the general solution of this equation is:

$$y = A \cos 25t + B \sin 25t, \qquad [1.6]$$

where y is the displacement of the body from the static rest position. An expression for the velocity is obtained by differentiating the above equation [1.6]. Hence,

$$\dot{y} = -25A \sin 25t + 25B \cos 25t. \qquad [1.7]$$

(a) For $t = 0$, $y = 0.1$ m.

From equation [1.6],

$$0.1 = A + 0$$

and

$$A = 0.1.$$

As the body was initially at rest, for $t = 0$, $\dot{y} = 0$. From equation [1.7],

$$0 = 0 + 25B$$

and

$$B = 0.$$

The expression for the displacement is finally:

$$y = 0.1 \cos 25t.$$

Comparing the above expression for the displacement with equation [1.5] one can deduce that the amplitude is 0.1 m and the phase angle is 0.

(b) $t = 0$, $y = 0$ because the body is not displaced during the application of the impulse (remember an impulse exists for a very short time).

From equation [1.6],

$$0 = A + 0$$

and

$$A = 0.$$

The integration of Newton's second law with respect to time leads to the relation:

Impulse = change in momentum

Hence,

$$10 = 4(\dot{y}(0) - 0),$$

where $\dot{y}(0)$ is the velocity immediately after the impact. Therefore, $\dot{y}(0) = 2.5$ m/s.

From equation [1.7], for $t = 0$

$$2.5 = 25B$$

and

$$B = 0.1.$$

Therefore,

$$y = 0.1 \sin 25t$$

or

$$y = 0.1 \cos (25t - \pi/2).$$

In this case (with the cosine) the amplitude is still 0.1 m, and by comparing with equation [1.5] it can be seen that the phase angle is 90°. The displacement discussed in part (b) lags by a phase angle of 90° the displacement obtained in part (a), as shown in Figure 1.5.

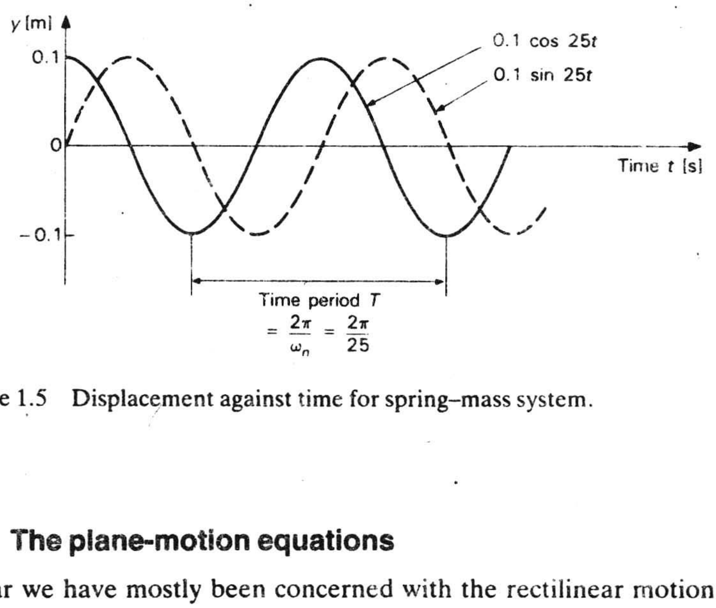

Figure 1.5 Displacement against time for spring–mass system.

1.6 The plane-motion equations

So far we have mostly been concerned with the rectilinear motion of a body on a spring. Many problems in this book will deal with plane motion or motion in two dimensions. The three equations that describe the dynamics of a rigid body in a plane (xy) are derived from Newton's laws of motion and are:

$$\Sigma F_x = m\ddot{x}_G \text{ } (\Sigma F_x\text{, force components in } x \text{ direction}) \quad [1.8]$$

$$\Sigma F_y = m\ddot{y}_G \text{ } (\Sigma F_y\text{, force components in } y \text{ direction}) \quad [1.9]$$

$$\Sigma M_G = I_G\ddot{\theta} \text{ } (\Sigma M_G\text{, moments about the rotation axis through the centre of mass G}). \quad [1.10]$$

where the body has mass m and moment of inertia I_G about the rotation axis through the centre of mass G (perpendicular to the xy plane). The moments are about the same axis. Note that the term rotation axis here is defined as an axis perpendicular to the plane of motion of the rigid body. The position of the centre of mass G of the body has the co-ordinates (x_G, y_G). The components of acceleration of G in the x and y directions are $\ddot{x}_G$ and $\ddot{y}_G$, respectively. The angular acceleration of the body is $\ddot{\theta}$. Further details on these equations and their applications may be obtained from books on mechanics such as *Dynamics* by J. L. Meriam (2nd edn), Wiley 1975, or *Principles of Engineering Mechanics* by H. R. Harrison and T. Nettleton, Edward Arnold 1978. When there is rotation of a rigid body about a fixed axis through point O one may take moments about the fixed rotation axis, so the equation of motion is:

$$\Sigma M_O = I_O\ddot{\theta}, \tag{1.11}$$

where I_O is the moment of inertia of the rigid body about the axis of rotation through O. In fact equation [1.11] also applies when, e.g., the axis has a constant velocity.

The parallel-axes theorem is used to relate I_O and I_G.

$$I_O = I_G + md^2, \tag{1.12}$$

where d is the distance between the parallel axes through O and G.

In many problems it is useful to find the instantaneous centre of rotation or the instantaneous centre of zero velocity, C. For a wheel, that rolls without slipping, C occurs at the point of contact with the ground, as shown in Figure 1.6(a). Provided the centre of mass coincides with the geometric centre of the wheel the following equation is allowed:

$$\Sigma M_C = I_C\ddot{\theta}, \tag{1.13}$$

where I_C is the moment of inertia about the rotation axis through the instantaneous centre of rotation and M_C is a moment about the same axis. Equation [1.13] also applies in the case of a spool unwinding from a rope, where C is located as shown in Figure 1.6(b).

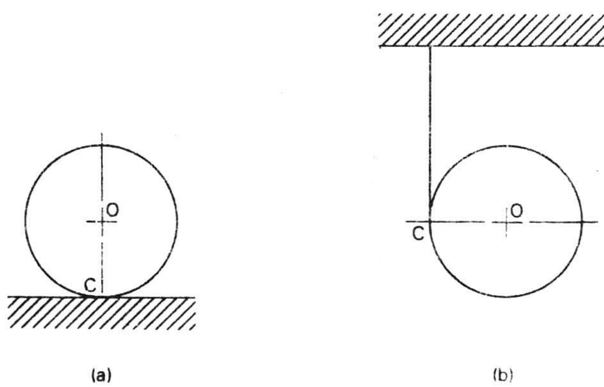

Figure 1.6 Location of instantaneous centre C for (a) rolling wheel, (b) spool unwinding.

Example 1.3
Natural frequency of a system with rolling

Determine the undamped natural frequency of small oscillations of the system shown in Figure 1.7(a). The uniform disc of mass m has two springs attached as indicated. Assume that the disc rolls without slipping.

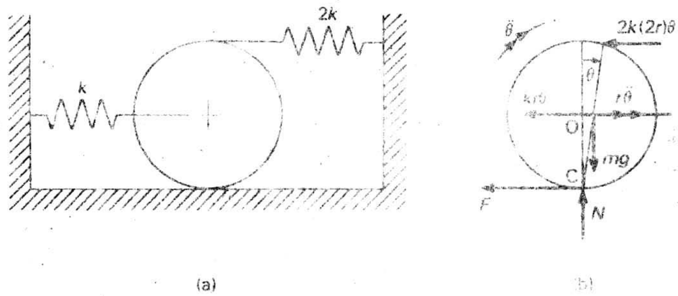

Figure 1.7 (a) Diagram for Example 1.3; (b) free-body diagram for disc in Example 1.3.

Solution As the wheel rolls on the horizontal surface the displacement of the centre of the wheel is $r\theta$. The spring of stiffness $2k$ is attached to a point on the top of the wheel. The horizontal component of the displacement of this point is approximately $2r\theta$ for small oscillations. The point on the wheel in contact with the surface is the instantaneous centre of rotation C. This point is shown on the free-body diagram of Figure 1.7(b). If we take moments about the axis of rotation through C the only forces that are involved are the two spring forces, provided the angle θ is small.

I_G for a uniform disc about the axis through its mass centre is $mr^2/2$; hence by equation [1.12]:

$$I_C = mr^2/2 + mr^2.$$

It should be noted that the centre of mass coincides with the geometrical centre of the disc. Therefore, equation [1.13] leads to:

$$-2r(4kr\theta) - r(kr\theta) = 1.5mr^2\ddot{\theta}$$

or

$$-9kr^2\theta = 1.5mr^2\ddot{\theta}$$

or

$$\ddot{\theta} + (6k/m)\theta = 0,$$

$$\omega_n = (6k/m)^{0.5},$$

and

$$f_n = \{1/(2\pi)\}(6k/m)^{0.5}.$$

The same result may be obtained by using equations [1.8] and [1.10]:

$$-F - kr\theta - 4kr\theta = mr\ddot{\theta},$$
$$+Fr - r(4kr\theta) = I_G\ddot{\theta}.$$

Multiplying the first equation by r and adding give:

$$-9kr^2\theta = 1.5mr^2\ddot{\theta},$$

as before.

Answer Undamped natural frequency is $\{\mathbf{1/(2\pi)}\}(\mathbf{6k/m})^{\mathbf{0.5}}$.

Example 1.4
Natural frequency of a torsional system

> One end of a solid steel shaft is fixed and a uniform steel disc is attached at its centre to the other end of the shaft. Dimensions of the shaft and disc are shown in Figure 1.8(a). The modulus of rigidity G for steel is 8×10^{10} Pa and the density of steel is 7.8×10^3 kg/m^3.
> Determine the natural frequency of torsional oscillations of the system.

Solution Moment of inertia I of the disc about the XX axis is $0.5mr^2$ or

$$I = 0.5(7.8 \times 10^3 \times \pi \times 0.090^2 \times 0.005)(0.090)^2$$
$$= 4.019 \times 10^{-3} \text{ kg m}^2.$$

Torsional stiffness s of the shaft is given by:

$$s = GI_p/l = 8 \times 10^{10}(\pi \times 0.012^4/32)/0.2$$
$$= 814.3 \text{ N m}.$$

The free-body diagram for the disc is shown in Figure 1.8(b). From equation [1.11]:

$$-s\theta = I\ddot{\theta}$$

or

$$\ddot{\theta} + \{814.3/(4.019 \times 10^{-3})\}\theta = 0.$$

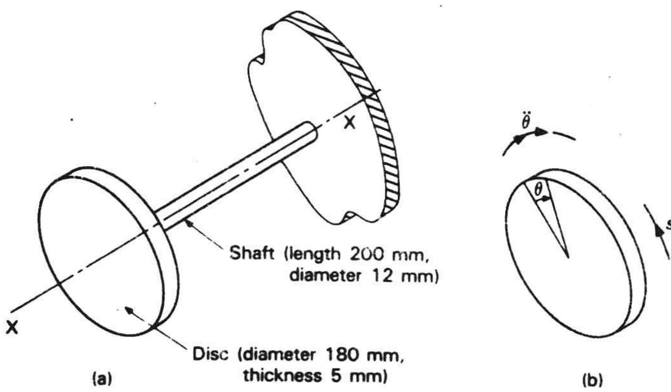

Figure 1.8 (a) Diagram for Example 1.4; (b) torque and angular acceleration on the disc.