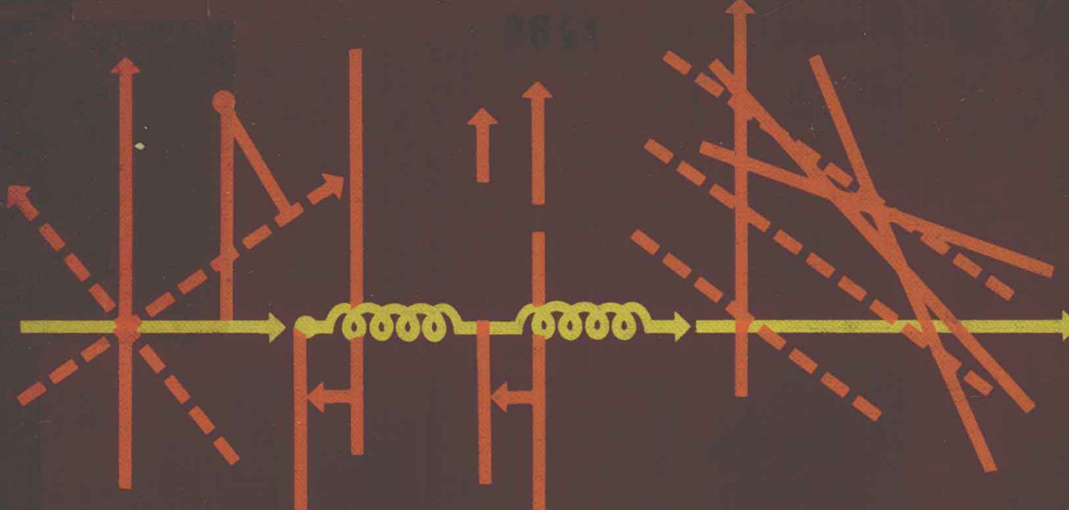
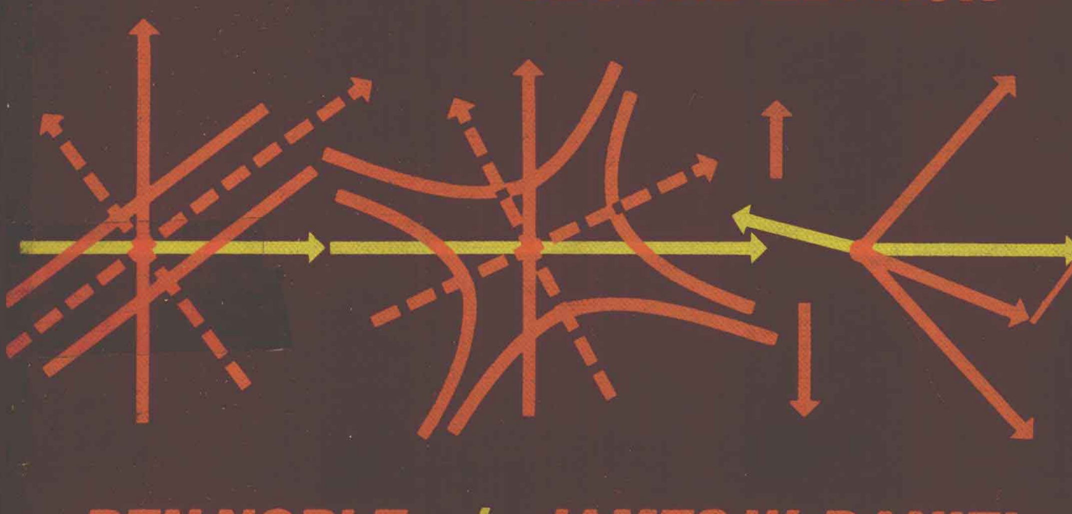


APPLIED LINEAR ALGEBRA

SECOND EDITION



BEN NOBLE / JAMES W. DANIEL

SECOND EDITION

APPLIED LINEAR ALGEBRA

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PREFACE

Linear algebra is an essential part of the mathematical toolkit required in the modern study of many areas in the behavioral, natural, physical, and social sciences, in engineering, in business, in computer science, and of course in pure and applied mathematics. The purposes of this book are to develop the fundamental concepts of linear algebra, emphasizing those concepts which are most important in applications, and to illustrate the applicability of these concepts by means of a variety of selected applications. Thus, while we present applications for illustrative and motivative purposes, our main goal is to present the *mathematics that can be applied.*

We have taken great care, however, to present the mathematical theory from a concrete point of view rather than from an abstract one. Because of this approach, we introduce the concrete manipulative matrix and vector algebra (Chapter 1) and elementary row operations (Chapter 3) well before the more abstract notions of vector spaces (Chapter 4) and linear transformations (Chapter 5). Similarly, eigenvalues, eigenvectors, and their associated canonical forms are presented (Chapter 8 and, in more detail, Chapters 9 and 10) as ways to simplify the study of linear transformations which perhaps describe the behavior of very complicated systems; our emphasis on these and other simplifying matrix decompositions (Chapter 9) and their applications (Chapters 9 and 10) is one of the unusual features of our approach.

In addition to presenting a number of applications briefly throughout the book both in the text and in the exercises, we have chosen to collect a variety of applications in Chapter 2 as well; there we assume familiarity with only the basic matrix algebra of Chapter 1. These applications serve to illustrate the ways in which linear algebra arises in applications and to motivate the later study of certain specific aspects of matrices. The efficient solution of systems of linear equations and the basic notions of linear programming can easily be viewed either as applications of linear algebra or as particular aspects of linear algebra which arise in an immense variety of concrete application areas; these important subjects are covered in Chapters 6 and 7.

It is important to note that the presentation throughout the book hinges on the use of elementary row operations; these manipulations as presented in Chapter 3 are used both as tools to develop theoretical notions (Sections 3.5, 3.6, 3.7, 4.5, 5.2, et cetera) and also as the basis of computational methods (Sections 3.2, 3.3, 4.3, 6.2, 7.1, 7.3, et cetera). It is *absolutely essential* that the student master the material on elementary row operations (Sections 3.2 and 3.4).

Although this book is considerably shorter than its predecessor (*Applied Linear Algebra* by Ben Noble; Prentice-Hall, 1969) and contains less peripheral material, it is still possible to teach a variety of courses from this one book; this places on the instructor a special responsibility carefully to select topics to be covered. To facilitate this selection we give some guidelines on the material to be included in various types of courses.

The *classical theoretical linear algebra course* will of course emphasize the mathematical theory; the reason for using our text for such a course is that in addition to the theory it provides applications, a few of which can be used for motivation and illustration. After covering Chapter 1 carefully and one or two topics from Chapter 2 very briefly, the instructor should then concentrate on the mathematical theory in Chapters 3, 4, and 5, covering the theorems and their proofs in detail. In the remaining time we recommend brief presentations of the main results of Chapters 8 and 11, primarily those results indicated in the text and the chapter introductions to be **Key Theorems**. If time permits, we encourage the inclusion of a small amount of applied material selected from Chapters 6, 7, 9, and 10.

The *introductory applied linear algebra course*, emphasizing the *applicability* of the fundamental theoretical results, is of course one of those for which this book was primarily intended. Many variations of such a course are possible, depending, for example, on what types of applications the instructor wishes to emphasize (for engineering? for social sciences? for statistics? et cetera) and on how much theorem-proving the instructor wishes to include (none? only selected **Key Theorems**? et cetera). The syllabus we now describe attempts to provide a good mix of these alternatives. We would cover Chapter 1 carefully and then discuss two applications selected from Chapter 2, say Sections 2.2

and 2.5, but depending on precisely what will be covered later. Next we proceed carefully through Chapter 3, usually "proving by example" rather than devoting too much time to the technical details of proofs, although the **Key Proof** of **Key Theorem 3.6** should be presented because of its importance; most of our time would be spent explaining the theorems indicated in the text to be **Key Theorems**. Chapters 4 and 5 should be covered somewhat more quickly but still carefully with most time spent motivating and explaining the **key** results. At this point the student should have the fundamental concepts and tools of linear algebra, except those involving eigenvalues. To allow time to absorb the preceding material we would very briefly skim Chapter 6, especially Sections 6.2 and 6.5, and quickly cover Sections 7.1, 7.2, and 7.3, perhaps skimming Section 7.5. Next we would complete the necessary conceptual background by covering Chapter 8, as usual concentrating on explanations of **key** results. The first three sections of Chapter 9 and the first four of Chapter 10 develop in detail the ideas outlined in Chapter 8, with Chapter 9 concentrating on what can be accomplished via unitary transformations while Chapter 10 treats the more general similarity transformations; given enough time we would present one of these two groups of sections, probably that in Chapter 9. We also would try to skim some of the advanced applications of these chapters (probably Sections 9.4, 9.5, 9.6, and 10.5), allowing time finally just to introduce the notions of Chapter 11.

It is difficult to settle on a specific format for presenting the material in Chapters 8, 9, and 10; different teachers may well prefer an organization different from that chosen by us. One can view these chapters as containing (1) basic concepts, (2) results on the structure of eigensystems, (3) results on matrix decompositions and canonical forms, and (4) applications. Since we felt that many teachers would concentrate on the simple and important case of normal matrices and unitary transformations, we chose to put (2), (3), and (4) for this case together in one chapter (Chapter 9) with the more general material in another (Chapter 10). A teacher who prefers to present first all the material on canonical forms, then the material on eigensystems, and finally the applications would be forced to jump about among the sections (e.g., 9.2, 9.4, 10.2, 10.3, 9.3, 10.4, 9.5, 9.6, 10.5, 10.6, 10.7) in order to arrange the presentation in that fashion. Our ordering was chosen to reduce such skipping about by the majority of the users of our book.

The above syllabus for an introductory applied linear algebra course includes material from every chapter and therefore quite obviously covers a large number of pages. Instructors must be careful not to become bogged down in technical details but rather to emphasize *concepts* and *techniques* by explaining and illustrating the **key** results and especially emphasizing how they can be used. Only Chapter 3 need be covered with great care, while some care should be exercised with Chapters 4 and 5; the remaining material moves quite quickly.

An *intermediate applied linear algebra course* can easily be built around this text for students who are familiar with the basic concepts of matrix and vector algebra and linear independence. Starting with a quick review and a study of two sections from Chapter 2, we recommend proceeding to a fairly quick coverage of Chapter 3, the last half of Chapter 4, and Chapter 5, depending on just how much linear algebra the students recall. We would cover Chapters 6 and 7 fairly briefly but with more detail than for the introductory applied course, cover Chapter 8 to fill in the holes in the students' knowledge here up to the level described for the introductory applied course, and then treat all of Chapters 9, 10, and 11, as usual, emphasizing ideas and techniques rather than proofs by concentrating on explanations of the **key** results. The obvious difference between this course and the introductory applied course is that the greater assumed background knowledge of the students in the intermediate course allows the material of Chapters 6 through 11 to replace the introductory material.

Several other types of applied courses can be created to fit the interests of the students or instructor. An audience oriented toward statistics and the social and behavioral sciences should probably be sure to include Chapter 7 in its entirety, Sections 8.5, 9.4, 9.5, 9.6, 9.7, and 10.5, and applications selected from Sections 2.2, 2.3, 2.5, and 2.6. An audience oriented more toward engineering might prefer Sections 6.1, 6.2, 6.3, 6.4, 8.4, 9.5, 10.3, 10.5, 10.6, 10.7, 11.5, and 11.6, and applications selected from Sections 2.4 and 2.6.

As indicated, we feel that this book has great flexibility to meet the special desires of knowledgeable instructors. To help those using the book for the first time, we have given, at the start of each chapter, brief statements of what we consider to be **key** results. These are intended to help both the student and the less experienced instructor recognize what are the fundamental conclusions of the chapter; any student understanding the **key** theorems of this text will have a solid foundation in linear algebra, matrix theory, and their applications. **key** material is indicated by the presence of a bold bullet (•) and by the use of the word **key** (as in **Key Theorem 3.6**).

Some remarks on notation are in order. Displayed equations are numbered consecutively within each chapter, so that (6.8) is the eighth numbered equation in Chapter 6. Likewise are corollaries, definitions, figures, lemmas, and theorems consecutively but *independently* numbered within each chapter; thus Definition 8.2 and Theorem 8.4 are respectively the second definition and the fourth theorem of Chapter 8, but Definition 8.2 *need not* precede Theorem 8.4 just because of its lower number. We have included an index to definitions and theorems to facilitate the reader's locating a referenced result. We have similarly numbered consecutively and independently the several hundred exercises indicated by Ex., some of which are presented along with solutions or suggested approaches. Thus Ex. 8.16 is essentially an example in that it is stated as an exercise but its solution is immediately presented in the text, Ex. 8.5 is a problem or exercise but with a solution (or partial solution or hint) that is given not in the text but rather in **Hints and Solutions** near the end of the book, and Ex. 8.6 is a problem or exercise whose solution is not presented or indicated anywhere in this book.

This book has a long history. It is a revision of the first author's earlier (1969) *Applied Linear Algebra*, whose history is detailed in the original's Preface (pp. ix–x); this new edition arose from a desire to create an edition that was more easily teachable and that presented a greater variety of applications at a less technically complex level. In this effort we have been greatly encouraged by Prentice-Hall in the persons of Ed Lugenbeel, John Hunger, Ken Cashman, and Cathy Brenn, whose cooperation and assistance we appreciate.

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CONTENTS

COMMON NOTATION

u, **x**, boldface small letters refer to vectors (column and row matrices), *1*

A, **P**, boldface capital letters refer to matrices other than vectors, *1*

k, α, italic Roman and Greek letters denote scalars, *5*

V, A, italic capital letters denote vector spaces or linear transformations, *108*, *151*

$\mathbf{A} = [a_{ij}]$, matrix, *1*

$\bar{\mathbf{A}} = [\bar{a}_{ij}]$, complex conjugate of **A**, *13*

$\mathbf{A}^T = [a_{ji}]$, transpose of **A**, *11*

$\mathbf{A}^H = [\bar{a}_{ji}]$, hermitian transpose of **A**, *13*

$\mathbf{A}^{-1}$, inverse of **A**, *17*

$\mathbf{A}_e$, $\mathbf{x}_e$, $\mathbf{c}_e$, extended vectors in linear programming, *220*

$\mathbf{A}^+$, generalized inverse of **A**, *337*

$\mathbf{A}_{ij}$, cofactor of matrix **A**, *199*

$\mathbf{C}^n$, n-dimensional space of complex column n-vectors, *111*

c(**A**), condition number of matrix **A**, *170*

D, matrix, usually diagonal, *387*

det **A**, determinant of **A**, *198*

$\mathbf{e}_i$, unit column vector, *22*

exp, exponential function: $\exp(x) = e^x$, *400*

$\mathbf{E}_{pq}$, $\mathbf{E}_p(c)$, $\mathbf{E}_{pq}(c)$, elementary matrices, *86*

$\mathbf{I}_m$, $m \times m$ unit matrix, *15*

$\mathbf{J}$, Jordan form, *361*

$\mathbf{J}_i$, Jordan block, *361*

$\mathbf{L}$, $\mathbf{U}$, lower and upper triangular matrices, respectively, *192*

$\mathbf{M}(A)$, matrix representation of linear transformation A, *154*

$\mathbf{P}$, $\mathbf{Q}$, unitary matrices, *282*

$\mathbf{R}$, upper- (or right-) triangular matrix, *317*

$\mathbf{R}_{ij}$, plane rotation matrix, *284*

$\mathbb{R}^n$, n-dimensional space of real column n-vectors, *111*

$\mathbf{T}$, tableau in linear programming, *225*

$\vec{u}$, geometrical vector, *103*

$\vec{u} \cdot \vec{v}$, dot product, *133*

$\mathbf{U}$, $\mathbf{V}$, unitary matrices, *282*

$\mathbf{0}$, $m \times n$ matrix of zeros, *9*

λ, eigenvalue, or parameter in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{b}$, *257, 264*

λ_i, $\mathbf{x}_i$, $i = 1, \ldots, n$, eigenvalues and corresponding eigenvectors of matrix, *264*

$\mathbf{\Lambda}$, diagonal matrix whose ith diagonal element is λ_i, *272*

$\rho(\mathbf{x})$, spectral radius, Rayleigh quotient, *389, 431*

$(\mathbf{u}, \mathbf{v})$, inner product, *134*

$(\mathbf{x}, \mathbf{Ax})$, quadratic form, *419*

$\|\mathbf{x}\|$, $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$, vector norms, *131*

$\|\mathbf{A}\|$, $\|\mathbf{A}\|_1$, $\|\mathbf{A}\|_2$, $\|\mathbf{A}\|_\infty$, matrix norms, *164, 165*

$\|\mathbf{A}\|_F$, Frobenius norm of a matrix, *328*

$>, \geq, <, \leq$, applied to matrices, *53, 220*

$\mathbf{\Sigma}$, diagonal $m \times n$ matrix of singular values, *325*

σ_i, singular value, *325*

CHAPTER ONE

MATRIX ALGEBRA

All of the material in this first chapter is fundamental: its goal is to introduce matrices and those basic algebraic manipulations which the student must thoroughly understand before proceeding. It is important to practice the addition and multiplication of matrices until these operations become automatic; the examples illustrating Theorem 1.3 will be helpful in this regard. Theorem 1.8 is a **key theorem,** providing the basis for later computational methods; the proofs of Theorems 1.5 and 1.8 have been labeled **key proofs** because they illustrate generally useful arguments.

1.1 INTRODUCTION

From an elementary point of view, matrices provide a convenient tool for systematizing laborious algebraic and numerical calculations. We define a *matrix* to be simply a set of (real or complex) numbers arranged in a rectangular array. Thus

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} x-a, & 4+b, & 1 \\ -2, & y, & -4 \end{bmatrix}, \quad [2, b], \quad [6] \qquad (1.1)$$

are matrices. The separate numbers in a given array are called the *elements* of the matrix, and these are, in general, completely independent of each other. The commas separating the elements, as in the above examples, will be omitted if there is no risk of confusion.

The general matrix consists of mn numbers arranged in m rows and n columns, giving the following $m \times n$ (read "m by n") array:

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{1.2}$$

The symbol a_{ij} denotes the number in the ith row and the jth column of the array:

$$\begin{array}{c} \quad j\text{th column} \\ i\text{th row} \begin{bmatrix} & \cdot & \\ & \cdot & \\ & \cdot & \\ \cdots & a_{ij} & \cdots \\ & \cdot & \\ & \cdot & \\ & \cdot & \end{bmatrix}. \end{array}$$

Thus if $\mathbf{A}$ is the second array in (1.1), then $a_{11} = x - a$, $a_{23} = -4$, etc. We often refer to a_{ij} as the (i, j) element of $\mathbf{A}$. We shall consider matrices whose elements can be real or complex numbers. The notation $[a_{ij}]$ is often convenient since it indicates that the general element of the matrix is a_{ij}. The subscript i runs from 1 to m and j from 1 to n. A comma is used to separate subscripts if there is any risk of confusion—e.g., $a_{p+q,\, r+s}$. We often consider a 1×1 matrix $[p]$ to be identical with the number p, although some care must be exercised in this regard as we shall see in Ex. 1.8.

The use of matrices allows us to consider an array of many numbers as a single object and to denote the array by a single symbol. Relationships between the large sets of numbers arising in applications can then be expressed in a clear and concise way. The more complicated the problem, the more useful matrix symbolism proves to be. In addition, however, as we have seen so often in the history of mathematics, a device that at first sight may appear to be mainly a notational convenience turns out to have extensive ramifications. The systematic application of matrices provides insights that could not be obtained as easily (if at all) by other methods.

1.2 EQUALITY, ADDITION, AND MULTIPLICATION BY A SCALAR

In the previous section, matrices were defined to be rectangular arrays of numbers. In order to work with these arrays, we need to specify rules for comparing and combining matrices. In particular, for matrices, we must develop rules corresponding to those governing the equality, addition, subtraction, multiplication, and division of ordinary numbers. We now state these rules, without attempting to provide any motivation apart from noting that they turn out to be precisely the rules required to deal with arrays of numbers that occur in applications and computational problems. This will be amply illustrated later.

DEFINITION 1.1. The matrices **A** and **B** are said to be *equal* if and only if:

(a) **A** and **B** have the same number of rows and the same number of columns.
(b) All corresponding elements are equal—i.e.,

$$a_{ij} = b_{ij} \qquad (\text{all } i, j).$$

We now consider the addition of matrices.

DEFINITION 1.2. Two matrices can be added if and only if they have the same number of rows and the same number of columns. In this case the *sum* of two $m \times n$ matrices **A** and **B** is the $m \times n$ matrix **C** such that any element of **C** is the sum of the corresponding elements in **A** and **B**, that is, if a_{ij}, b_{ij}, c_{ij} denote the general elements of **A**, **B**, **C**, respectively, then

$$a_{ij} + b_{ij} = c_{ij} \qquad (\text{all } i, j),$$

or, in matrix notation,

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [c_{ij}] = \mathbf{C}. \tag{1.3}$$

For example,

$$\begin{bmatrix} x+1 & -1 \\ 2 & y-1 \end{bmatrix} + \begin{bmatrix} -1 & a \\ b & 1 \end{bmatrix} = \begin{bmatrix} x & a-1 \\ 2+b & y \end{bmatrix}.$$

Since the sum of two matrices is formed by simply adding corresponding elements, it is clear that the rules governing the addition of matrices are precisely the same as those governing the addition of ordinary numbers.