Lecture Notes in Mathematics

1566

B. Edixhoven J.-H. Evertse (Eds.)

Diophantine Approximation and Abelian Varieties

Introductory Lectures

Springer-Verlag

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

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Preface

From April 12 to April 16, 1992, the instructional conference for Ph.D-students "Diophantine approximation and abelian varieties" was held in Soesterberg, The Netherlands. The intention of the conference was to give Ph.D-students in number theory and algebraic geometry (but anyone else interested was welcome) some acquaintance with each other's fields. In this conference a proof was presented of Theorem I of G. Faltings's paper "Diophantine approximation on abelian varieties", Ann. Math. 133 (1991), 549–576, together with some background from diophantine approximation and algebraic geometry. These lecture notes consist of modified versions of the lectures given at the conference.

We would like to thank F. Oort and R. Tijdeman for organizing the conference, the speakers for enabling us to publish these notes, C. Faber and W. van der Kallen for help with the typesetting and last but not least the participants for making the conference a successful event.

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Introduction

Although diophantine approximation and algebraic geometry have different roots, today there is a close interaction between these fields. Originally, diophantine approximation was the branch in number theory in which one deals with problems such as approximation of irrational numbers by rational numbers, transcendence problems such as the transcendence of e or π, etc. There are some very powerful theorems in diophantine approximation with many applications, among others to certain classes of diophantine equations. It turned out that several results from diophantine approximation could be improved or generalized by techniques from algebraic geometry. The results from diophantine approximation which we discuss in detail in these lecture notes are Roth's theorem, which states that for every algebraic number α and for every $\delta > 0$ there are only finitely many $p, q \in \mathbb{Z}$ with $|\alpha - p/q| < |q|^{-2-\delta}$, and a powerful higher dimensional generalization of this, the so-called Subspace theorem of W.M. Schmidt. Here, we would like to mention the following consequence of the Subspace theorem, conjectured by S. Lang and proved by M. Laurent: let Γ be the algebraic group $(\overline{\mathbb{Q}}^*)^n$, endowed with coordinatewise multiplication, V a subvariety of Γ, not containing a translate of a positive dimensional algebraic subgroup of Γ, and G a finitely generated subgroup of Γ; then $V \cap \Gamma$ is finite.

We give a brief overview of the proof of Roth's theorem. Suppose that the equation $|\alpha - p/q| < q^{-2-\delta}$ has infinitely many solutions $p, q \in \mathbb{Z}$ with $q > 0$. First one shows that for sufficiently large m there is a polynomial $P(X_1, \ldots, X_m)$ in $\mathbb{Z}[X_1, \ldots, X_m]$ with "small" coefficients and vanishing with high order at $(\alpha, \ldots, \alpha)$. Then one shows that P cannot vanish with high order at a given rational point $x = (p_1/q_1, \ldots, p_n/q_n)$ satisfying certain conditions. This non-vanishing result, called Roth's Lemma, is the most difficult part of the proof. From the fact that $|\alpha - p/q| < q^{-2-\delta}$ has infinitely many solutions it follows that one can choose x such that $|\alpha - p_n/q_n| < q_n^{-2-\delta}$ for n in $\{1, \ldots, m\}$. Then for some small order partial derivative P_i of P we have $P_i(x) \neq 0$. But $P_i(x)$ is a rational number with denominator dividing $a := q_1^{d_1} \cdots q_m^{d_m}$, where $d_j = \deg_{X_j}(P_i)$. Hence $|P_i(x)| \geq 1/a$. On the other hand, P_i is divisible by a high power of $X_j - \alpha$ and $|p_j/q_j - \alpha|$ is small for all j in $\{1, \ldots, m\}$. Hence $P_i(x)$ must be small. One shows that in fact $|P_i(x)| < 1/a$ and thus one arrives at a contradiction.

Algebraic geometry enables one to study the geometry of the set of solutions (e.g., over an algebraically closed field) of a set of algebraic equations. The geometry often predicts the structure of the set of arithmetic solutions (e.g., over a number field) of these algebraic equations. As an example one can mention Mordell's conjecture, which was proved by G. Faltings in 1983 [21]. Several results of this type have been proved by combining techniques from algebraic geometry with techniques similar to those used in the proof of Roth's theorem. Typical examples are the Siegel-Mahler finiteness theorem for integral points on algebraic curves and P. Vojta's recent proof of Mordell's conjecture.

In these lecture notes, we study the proof of the following theorem of G. Faltings ([22], Thm. I), which is the analogue for abelian varieties of the result for $(\overline{\mathbb{Q}}^*)^n$

mentioned above, and which was conjectured by S. Lang and by A. Weil:

Let A be an abelian variety over a number field k and let X be a subvariety of A which, over some algebraic closure of k, does not contain any positive dimensional abelian variety. Then the set of rational points of X is finite.

(Note that this theorem is a generalization of Mordell's conjecture.) The proof of Faltings is a higher dimensional generalization of Vojta's proof of Mordell's conjecture and has some similarities with the proof of Roth's theorem. Basically it goes as follows. Assume that $X(k)$ is infinite. First of all one fixes a very ample symmetric line bundle $\mathcal{L}$ on A, and norms on $\mathcal{L}$ at the archimedian places of k. Let m be a sufficiently large integer. There exists $x = (x_1, \dots, x_m)$ in $X^m(k)$ satisfying certain conditions (e.g., the angles between the x_i with respect to the Néron-Tate height associated to $\mathcal{L}$ should be small, the quotient of the height of x_{i+1} by the height of x_i should be big for $1 \leq i < m$ and the height of x_1 should be big). Instead of a polynomial one then constructs a global section f of a certain line bundle $\mathcal{L}(\sigma-\varepsilon, s_1, \dots, s_m)^d$ on a certain model of X^m over the ring of integers R of k. This line bundle is a tensor product of pullbacks of $\mathcal{L}$ along maps $A^m \to A$ depending on $\sigma-\varepsilon$, the s_i and on d; in particular, it comes with norms at the archimedian places. By construction, f has small order of vanishing at x and has suitably bounded norms at the archimedian places of k. Then one considers the Arakelov degree of the metrized line bundle $x^*\mathcal{L}(\sigma-\varepsilon, s_1, \dots, s_m)^d$ on Spec(R); the conditions satisfied by the x_i give an upper bound, whereas the bound on the norm of f at the archimedian places gives a lower bound. It turns out that one can choose the parameters ε, σ, the s_i and d in such a way that the upper bound is smaller than the lower bound.

We mention that the construction of f is quite involved. Intersection theory is used to show that under suitable hypotheses, the line bundles $\mathcal{L}(-\varepsilon, s_1, \dots, s_m)^d$ are ample on X^m. A new, basic tool here is the so-called Product theorem, a strong generalization by Faltings of Roth's Lemma.

On the other hand, Faltings's proof of Thm. I above is quite elementary when compared to his original proof of Mordell's conjecture. For example, no moduli spaces and no l-adic representations are needed. Also, the proof of Thm. I does not use Arakelov intersection theory. Faltings's proof of Thm. I in [22] seems to use some of it, but that is easily avoided. The Arakelov intersection theory in [22] plays an essential role in the proof of Thm. II of [22], where one needs the notion of height not only for points but for subvarieties; we do not give details of that proof. The only intersection theory that we need concerns intersection numbers obtained by intersecting closed subvarieties of projective varieties with Cartier divisors, so one does not need the construction of Chow rings. The deepest result in intersection theory needed in these notes is Kleiman's theorem stating that the ample cone is the interior of the pseudo-ample cone. Unfortunately, we will have to use the existence and quasi-projectivity of the Néron model over Spec(R) of A in the proof of Lemma 3.1 of Chapter XI; a proof of that lemma avoiding the use of Néron models would significantly simplify the proof of Thm. I. We believe that for someone with a basic knowledge of algebraic geometry, say Chapters II and III of [27], everything in these notes except for the use of Néron models is not hard to understand. In the case where X is a curve, i.e., Mordell's conjecture, the proof of Thm. I can be considerably simplified; this was done by E. Bombieri in [9].

Let us now describe the contents of the various chapters. Chapter I gives an overview of several results and conjectures in diophantine approximation and arithmetic geometry. After that, the lecture notes can be divided in three parts.

The first of these parts consists of Chapters II–IV; some of the most important results from diophantine approximation are discussed and proofs are sketched of Roth's theorem and of the Subspace theorem.

The second part, which consists of Chapters V–XI, deals with the proof of Thm. I above. Chapters V and VII provide the results needed of the theory of height functions and of intersection theory, respectively. Chapter VIII contains a proof of the Product theorem. This theorem is then used in Chapter IX in order to prove the ampleness of certain $\mathcal{L}(-\varepsilon, s_1, \ldots, s_m)^d$. Chapter X gives a proof of Faltings's version of Siegel's Lemma. Chapter XI finally completes the proof of Thm. I. Chapter VI gives some historical background on how D. Mumford's result on the "widely spacedness" of rational points of a curve of genus at least two over a number field lead to Vojta's proof of Mordell's conjecture.

The third part consists of Chapters XII and XIII. Chapter XII gives an application of Thm. I to the study of points of degree d on curves over number fields. Chapter XIII discusses a generalization by Faltings of Thm. I, which was also conjectured by Lang.

Terminology and Prerequisites

In these notes it will be assumed that the reader is familiar with the basic objects of elementary algebraic number theory, such as the ring of integers of a number field, its localizations and completions at its maximal ideals, and the various embeddings in the field of complex numbers. The same goes more or less for algebraic geometry. To understand the proof of Faltings's Thm. I the reader should be familiar with schemes, morphisms between schemes and cohomology of quasi-coherent sheaves of modules on schemes. In order to encourage the reader, we want to mention that Hartshorne's book [27], especially Chapters II, §§1–8 and III, §§1–5 and §§8–10, contains almost all we need. The two most important exceptions are Kleiman's theorem on the ample and the pseudo-ample cones (see Chapter VII), for which one is referred to [28], and the existence and quasi-projectivity of Néron models of abelian varieties (used in Chapter XI), for which [11] is an excellent reference. At a few places the "GAGA principle" (see [27], Appendix B) and some algebraic topology of complex analytic varieties are used. A less important exception is the theorem of Mordell-Weil, a proof of which can for example be found in Manin's [52], Appendix II, or in [70]; Chapter V of these notes contains the required results on heights on abelian varieties. Almost no knowledge concerning abelian varieties will be assumed. By definition an abelian variety over a field k will be a commutative projective connected algebraic group over k. We will use that the associated complex analytic variety of an abelian variety over $\mathbb{C}$ is a complex torus.

Since these notes are written by various authors, the terminologies used in the various chapters are not completely the same. For example, Chapter I uses a normalization of the absolute values on a number field which is different from the normalization used by the other contributors; the reason for this normalization in Chapter I is clear, since one no longer has to divide by the degree of the number field in question to define the absolute height, but it has the disadvantage that the absolute value no longer just depends on the completion of the number field with respect to the absolute value. Another example is the notion of variety. If k is a field, then by a *(algebraic) variety (defined) over* k one can mean an integral, separated k-scheme of finite type; but one can also mean the following: an *(absolutely irreducible) affine variety (defined) over* k is an irreducible Zariski closed subset in some affine space K^n (K a fixed algebraically closed field containing k) defined by polynomials with coefficients in k, and a *(absolutely irreducible) variety (defined) over* k is an object obtained by glueing affine varieties over k with respect to glueing data given again by polynomials with coefficients in k. As these two notions are (supposed to be) equivalent, no (serious) confusion should arise.

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Chapter I

Diophantine Equations and Approximation

by Frits Beukers

1 Heights

Let F be an algebraic number field. The set of valuations on F is denoted by M_F. Let $|.|_v$, or v in shorthand, be a valuation of F. Denote by F_v the completion of F with respect to v. If F_v is $\mathbb{R}$ or $\mathbb{C}$ we assume that v coincides with the usual absolute value on these fields. When v is a finite valuation we assume it normalised by $|p|_v = 1/p$ where p is the unique rational prime such that $|p|_v < 1$. The *normalised* valuation $||.||_v$ is defined by

$$||x||_v = |x|_v^{[F_v:\mathbb{Q}_p]/[F:\mathbb{Q}]}$$

with the convention that $p = \infty$ when v is archimedean and $\mathbb{Q}_\infty = \mathbb{R}$. For any non-zero $x \in F$ we have the *product formula*

$$\prod_v ||x||_v = 1. \tag{1.1}$$

Let L be any finite extension of F. Then any valuation w of L restricted to F is a valuation v of F. We have for any $x \in F$ and $v \in M_F$,

$$||x||_v = \prod_{w|v} ||x||_w \tag{1.2}$$

where the product is over all valuations $w \in M_L$ whose restriction to F is v. The ***absolute multiplicative height*** of x is defined by

$$H(x) = \prod_v \max(1, ||x||_v).$$

It is a consequence of (1.2) that $H(x)$ is independent of the field F which contains x. The *absolute logarithmic height* is defined by

$$h(x) = \log H(x).$$

Let $\mathbb{P}^n$ be the n-dimensional projective space and let $P \in \mathbb{P}^n(F)$ be an F-rational point with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. We define the *projective (absolute) height* by

$$h(P) = \sum_v \log \max(||x_0||_v, ||x_1||_v, \ldots, ||x_n||_v).$$

Again, $h(P)$ is independent of the field F containing P. Therefore the projective height can be considered as a function on $\mathbb{P}^n(\overline{F})$. Notice that the height $h(x)$ of a number coincides with the projective height of the point $(1:x) \in \mathbb{P}^1$. The projective height has the fundamental property that, given h_0, there are only finitely many $P \in \mathbb{P}^n(F)$ such that $h(P) < h_0$.

Let V be a non-singular projective variety defined over F. Let $\phi : V \hookrightarrow \mathbb{P}^N$ be a projective embedding also defined over F. On $V(\overline{F})$, the $\overline{F}$-rational points of V, we take the restriction of the projective height as a height function and denote it by h_ϕ. In general the construction of heights on V runs as follows. First, let D be a very ample divisor. That is, letting $f_0, f_1, \ldots, f_n$ be a basis of the space of all rational functions f defined over F with $(f) \geq -D$, the map $\phi : V \to \mathbb{P}^n$ given by $P \mapsto (f_0(P), f_1(P), \ldots, f_n(P))$ is a projective embedding. The height h_D is then simply defined as h_ϕ. If D_1, D_2 are two linearly equivalent very ample divisors, then $h_{D_1} - h_{D_2}$ is known to be a bounded function on $V(\overline{F})$.

Now let D be any divisor. On a non-singular projective variety one can always find two very ample divisors X, Y such that $D + Y = X$. Define $h_D = h_X - h_Y$. Again, up to a bounded fuction, h_D is independent of the choice of X and Y.

We summarize this height construction as follows.

1.3 Theorem. *There exists a unique homomorphism*

$$\text{linear divisor classes} \quad \to \quad \begin{array}{c}\text{real valued functions on } V(\overline{F}) \\ \text{modulo bounded functions}\end{array}$$

denoted by $c \mapsto h_c + O(1)$ *such that: if* c *contains a very ample divisor, then* h_c *is equivalent to the height associated with a projective embedding obtained from the linear system of that divisor.*

We also recall the following theorem.

1.4 Theorem. *Let* c *be a linear divisor class which contains a positive divisor* Z*. Then*

$$h_c(P) \geq O(1)$$

for all $P \in V(\overline{F})$, $P \notin supp(Z)$.

For the proof of the two above theorems we refer to Lang's book [36], Chapter 4.

Finally, following Lang, we introduce the notion of *pseudo ample divisor*, not to be confused with the pseudo ample cone. A divisor D on a variety V is said to be pseudo ample if some multiple of D generates an embedding from some non-empty Zariski open part of V into a locally closed part of projective space. One easily sees that there exists a proper closed subvariety W of V such that, given h_0, the inequality $h_D(P) < h_0$ has only finitely many solutions in $V(F) - W$.

2 The Subspace Theorem

For the sake of later comparisons we shall first state the so-called *Liouville inequality*.

2.1 Theorem (Liouville). *Let F be an algebraic number field and L a finite extension. Let S be a finite set of valuations and extend each $v \in S$ to L. Then, for every $\alpha \in L$, $\alpha \neq 0$ we have*

$$\prod_{v\in S} ||\alpha||_v \geq \frac{1}{H(\alpha)^{[L:F]}}.$$

Proof. Let us assume that $||\alpha||_v < 1$ for every $v \in S$. If not, we simply reduce the set S. Let S_L be the finite set of valuations on L which are chosen as extension of v on F. Using the product formula we find that

$$\begin{aligned}\prod_{w\in S_L} ||\alpha||_w &= \prod_{w\notin S_L} ||\alpha||_w^{-1} \\ &\geq \prod_{w\notin S_L} \max(1, ||\alpha||_w)^{-1} \\ &\geq \prod_{w\in M_L} \max(1, ||\alpha||_w)^{-1} = \frac{1}{H(\alpha)}.\end{aligned}$$

The proof is finished by noticing that

$$||\alpha||_v = ||\alpha||_w^{[L:F]/[L_w:F_v]} \geq ||\alpha||_w^{[L:F]}$$

□

Liouville applied more primitive forms of this inequality to obtain lower bounds for the approximation of fixed algebraic numbers by rationals. In our more general setting, let α be a fixed algebraic number of degree d over F. Then it is a direct consequence of the previous theorem that

$$\prod_{v\in S}(||x-\alpha||_v) > \frac{c(\alpha)}{H(x)^d} \tag{2.2}$$

for every $x \in F$ with $x \neq 0$. Here $c(\alpha)$ is a constant which can be taken to be $(2H(\alpha))^{-d}$. Using such an inequality Liouville was the first to prove the existence of transcendental numbers by constructing numbers which could be approximated by rationals much faster than algebraic numbers. In 1909 A. Thue provided the first non-trivial improvement over (2.2) which was subsequently improved by C.L. Siegel (1921), F. Dyson (1948) and which finally culminated in Roth's theorem, proved around 1955. The theorem we state here is a version by S. Lang which includes non-archimedean valuations, first observed by Ridout, and a product over different valuations.

2.3 Theorem (Roth). *Let F be an algebraic number field and S a finite set of valuations of F. Let $\epsilon > 0$. Let $\alpha \in \overline{\mathbb{Q}}$ and extend each v to $F(\alpha)$. Then*

$$\prod_{v\in S}(||x-\alpha||_v) < \frac{1}{H(x)^{2+\epsilon}}$$

has only finitely many solutions $x \in F$.

A proof of Roth's original theorem can be found in Chapter III of these notes. Around 1970 W.M. Schmidt extended Roth's techniques in a profound way to obtain a simultaneous approximation result. Again the version we state here is a later version which follows from work of H.P. Schlickewei.

$\Sigma \cap \Gamma = \{(x,\xi) \in \Gamma; r_i(x,\xi) = 0,\ i \leq d\}$,

$\{r_i, r_j\} = 0$, where $\{f, g\}$ is the Poisson bracket of f and g,

the differentials dr_i, $i \leq d$, and the canonical one-form $\sum \xi_i dx_i$ are linearly independent at all points from Σ which lie in Γ.

An important notion in this context is that of the bicharacteristic leaves of Σ. By definition these are the $d-$dimensional submanifolds in Σ which have as tangent vectors the Hamiltonian vectorfields associated with the r_i. As is standard, it follows from the assumptions on Σ that we can find homogeneous real-analytic canonical coordinates such that in a conic neighborhood of (x^0, ξ^0),

$$\Sigma = \{(x,\xi); \xi_i = 0 \text{ for } i \leq d\}. \tag{1.1.2}$$

In these coordinates, the bicharacteristic leaves are of course just of form $x_i = constant$, $\xi_i = constant$, $i = d+1, ..., n$. We shall also often set in this situation $\xi' = (\xi_1, ..., \xi_d)$, so that Σ becomes $\xi' = 0$ in some suitable local coordinate patch. Correspondingly, we set

$$x' = (x_1, \ldots, x_d), x'' = (x_{d+1}, \ldots, x_n), \xi'' = (\xi_{d+1}, \ldots, \xi_n).$$

(Related notations shall also be considered later on.)

An interesting case for results on propagation of singularities is here when p_m is transversally elliptic to Σ. By this we mean that if we fix $(y, \eta) \in \Sigma$, then we can find a conic neighborhood Γ of (y, η) and some c_1, c_2 so that

$$d_\Sigma^s(x,\xi)|\xi|^{m-s} \leq c_1|p_m(x,\xi)| \leq c_2 d_\Sigma^s(x,\xi)|\xi|^{m-s} \text{ if } (x,\xi) \in \Gamma, \tag{1.1.3}$$

where d_Σ is some homogeneous distance function to Σ. Actually, Σ is then precisely the characteristic variety of p, so p_m vanishes of constant multiplicity on its characteristic variety. We have then the following classical result of Bony-Schapira [1]:

Theorem 1.1.1. *Let u be a solution of $p(x, D)u = 0$ and denote by $WF_A u$ the analytic wave front set of u. Then $WF_A u$ is a union of bicharacteristic leaves of Σ. (More explicitly, if L is a connected bicharacteristic leaf of Σ and if $(x^0, \xi^0) \in WF_A u$, then $L \subset WF_A u$.)*

The case when p_m is real-valued and $s = 1$ in (1.1.1), i.e. when p is of so-called real principal type, theorem 1.1.1 had of course been considered already by Hörmander [2] and Kashiwara (cf. Sato-Kawai-Kashiwara [1]) and has been the prototype of all results on propagation of microlocal singularities ever since. Note that in this case the bicharacteristic leaves are just the null bicharacteristic curves of p_m. (For related results for