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lecture notes in pure and applied mathematic

nonlinear functional analysis and differential equations

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Proceedings of the Michigan State University Conference

NONLINEAR FUNCTIONAL ANALYSIS AND DIFFERENTIAL EQUATIONS

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FOREWORD

In recent years there has been considerable research in nonlinear functional analysis and its applications to differential equations. Hence it seemed appropriate, in the spring of 1975, to plan a conference which would bring together workers in this field to exchange thoughts, lay out a view of the present state of the field, and explore areas of future research.

This conference was held June 9 to 12, 1975 at Michigan State University with R. Kannan, then a visitor at Michigan State, and J. D. Schuur as its directors. Its format was six one-hour lectures by the principal speaker, L. Cesari, who developed in depth some aspects of the general topic; nine one-hour lectures by other invited speakers, who reviewed different aspects of the general topic; and research reports by participating mathematicians. This volume contains the lectures of the principal speaker and of the one-hour speakers and constitutes therefore the proceedings of the conference.

For his topic, Professor Cesari chose "Functional Analysis, Nonlinear Differential Equations, and the Alternative Method." He emphasized points at present under investigation, in particular the use of Banach's fixed point theorem, a priori bounds and topological arguments, monotone operators, and Schauder's fixed point theorem.

The one-hour talks ranged from problems in bifurcation theory to measurability of solutions of differential equations to current aspects of degree theory.

In such a large and active area, a survey will necessarily be limited by the choice of topics and a cutting point in time. However, we hope that this volume will prove useful in furthering the aims of the conference.

The editors are grateful to the Michigan State University Mathematics Department and its chairman, Professor J.E. Adney, for financial support, advice, and encouragement.

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CONTENTS

FUNCTIONAL ANALYSIS, NONLINEAR DIFFERENTIAL EQUATIONS, AND THE ALTERNATIVE METHOD

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Section 1. The Alternative Method

1. Introduction. In this presentation of the alternative, or bifurcation method emphasis is given to certain points of recent research, in particular in connection with the use of Banach's fixed point theorem (nos. 6-19), a priori bounds and topological arguments (nos. 20-27), monotone operators (nos. 28-33), and Schauder's fixed point theorem (nos. 34-42).

We are interested in solving an operator equation $Ax = 0$ in a space X. We shall think of A as the difference $A = E - N$ of a linear operator E, not necessarily bounded, and of an operator N, not necessarily linear, and thus the equation takes the form

$$Ex = Nx, \quad x \in X.$$

If E^{-1} exists, or equivalently the null space of E is trivial, then this equation can be written as $x - E^{-1}Nx = 0$, or $(I + KN)x = 0$. This is a Hammerstein equation and very extensive work has been done in this direction. For references to the wide literature one can see [147].

Whenever KN is compact, concepts and methods of topological degree theory in function spaces are relevant, and we should refer to the work of Leray-Schauder [167], Rothe [184], and Nagumo [177, 178].

We are particularly interested in the case where E has a nontrivial null space, the case which is often mentioned as the "problem at resonance." Thus we shall not exclude that E may have a nonzero null space. This will be actually the most interesting case.

In the line of the bifurcation process of Poincaré [180], Lyapunov [171], and Schmidt [187] we should decompose the equation $Ex = Nx$ into a system of two equations, possibly in different spaces. Much work has been done in this direction, for which we refer to [191] and [192].

The injection of ideas of functional analysis in the last years has made the process a remarkably fine tool of analysis, particularly in the difficult "problem at resonance" in the usual terminology.

The general theory which has ensued, with all its variants and ramifications through the works of many authors, is often referred to as the bifurcation theory, or the alternative method ([6], [7], [65], and recent expositions, e.g., [21-23], [63]). It is being used in theoretical existence analysis of the solutions, in methods of successive approximations of the solutions, in estimating the error of approximate solutions, in problems of perturbations, and in problems of bifurcation of eigenvalues.

2. The Alternative Scheme. To be specific, let us consider an equation of the form

$$Ex = Nx, \tag{2.1}$$

where $E:\mathcal{D}(E) \to Y$ is a linear operator, and $N:\mathcal{D}(N) \to Y$ is an operator, not necessarily linear, both E and N have domains $\mathcal{D}(E)$, $\mathcal{D}(N)$ in a Banach space X and ranges $\mathcal{R}(E)$, $\mathcal{R}(N)$ in a Banach space Y, $\mathcal{D}(E) \cap \mathcal{D}(N) \neq \emptyset$.

Let us assume for a moment that the following holds (we shall see in no. 3 that these assumptions are quite natural, and they are also easily verified in the so-called selfadjoint case). Let us assume that there are projection operators $P:X \to X$, $Q:Y \to Y$ (that is, linear, bounded, idempotent, or $PP = P$, $QQ = Q$), and a linear operator H, which we shall consider as a partial inverse of E satisfying

$$H(I - Q)Ex = (I - P)x \quad \text{for all } x \in \mathcal{D}(E), \tag{k_1}$$

$$QEx = EPx \quad \text{for all } x \in \mathcal{D}(E), \tag{k_2}$$

$$EH(I - Q)Nx = (I-Q)Nx \quad \text{for all } x \in \mathcal{D}(E) \cap \mathcal{D}(N). \tag{k_3}$$

Under these assumptions, the equation $Ex = Nx$ is equivalent to the system of two equations

$$x = Px + H(I - Q)Nx, \tag{2.2}$$

$$Q(Ex - Nx) = 0. \tag{2.3}$$

Indeed, if (2.1) is satisfied, then by applying Q to equation (2.1) we obtain (2.3). By applying $H(I - Q)$ to (2.1) we obtain $H(I - Q)Ex = H(I - Q)Nx$, and by using (k_1) we obtain (2.2). Conversely, if (2.2) and (2.3) are satisfied, then, by applying E to (2.2) we have $Ex - EPx = EH(I - Q)Nx$, and by using (k_2), (k_3), also $Ex - QEx = (I - Q)Nx$, or $Ex - Nx = Q(Ex - Nx) = 0$.

Equations (2.2) and (2.3) are usually denoted as the auxiliary and the bifurcation equations, respectively.

Let $X_0 = PX$, $X_1 = (I - P)X$, $Y_0 = QY$, $Y_1 = (I - Q)Y$, so that X and Y have the decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$ (direct sums), $X_0 \cap X_1 = \{0\}$, $Y_0 \cap Y_1 = \{0\}$. The linear subspace of X of all elements $x \in X$ for which $Ex = 0$ is often denoted as the null space of E, or kernel of E, and denoted by $\ker E$. When there are elements $x \neq 0$ in X with $Ex = 0$, that is, $\ker E$ is not trivial, then we say that we have a resonance case. We shall always assume that $\ker E \subset X_0$. Also, we shall assume that $Y_1 \subset \mathcal{R}(E)$, $\mathcal{D}(H) = Y_1$, $\mathcal{R}(H) = \mathcal{D}(E) \cap X_1$. Thus $EHy = y$ for all $y \in Y_1$, $HEx = x$ for all $x \in \mathcal{D}(E) \cap X_1$. Often, we may assume $\ker E = X_0$, $Y_1 = \mathcal{R}(E)$. In this particular situation the bifurcation equation reduces to $QNx = 0$.

If we write $x^* = Px$, then the auxiliary equation can always be written in the form of a fixed point statement:

$$x = Tx, \qquad Tx = x^* + H(I - Q)Nx, \tag{2.4}$$

where T is easily seen to map the fiber $P^{-1}x^*$ into itself. The auxiliary equation can also be written in the form of a Hammerstein equation:

$$x + KNx = x^*, \qquad K = -H(I - Q). \tag{2.5}$$

Finally the auxiliary equation, written in the equivalent form

$$Ex - Nx = Q(Ex - Nx),$$

is often called the relaxed equation, or relaxed problem.

There are certain rather general situations (N locally Lipschitzian, N monotone, etc.) under which the operators P, Q, H can be so chosen that the ensuing map T is a contraction map on suitable sets, or has other suitable