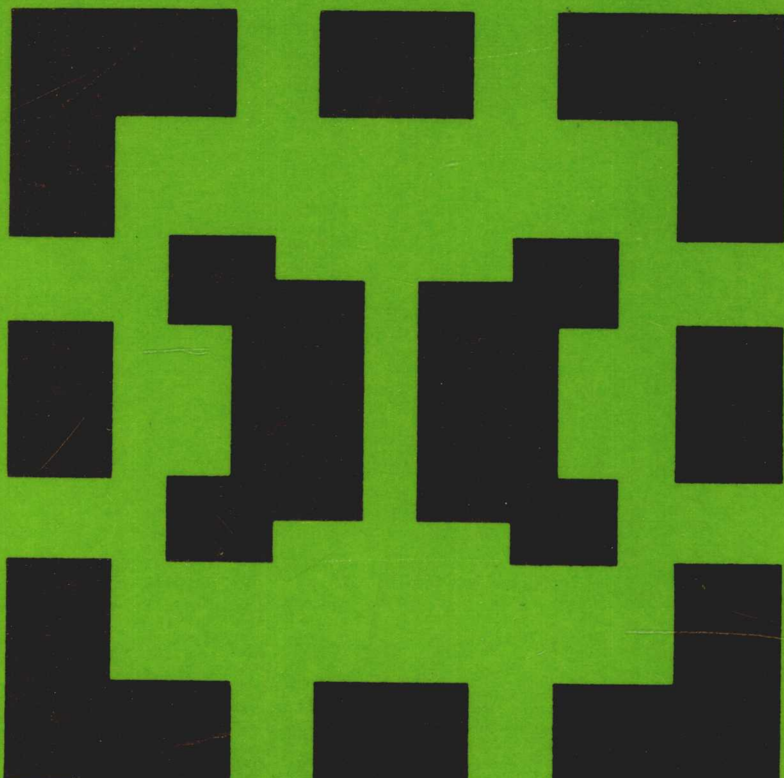
Mathematics and Its Applications

Gerhard Preuss

Theory of Topological Structures

An Approach to Categorical Topology



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Mathematics and Its Applications

SERIES EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Topology, a relatively new branch of mathematics, tries to capture such ideas as nearness and limits. It is of course immensely useful in virtually all branches of pure and applied mathematics including algebra and logic which, at first sight, seem far removed from the ideas at the basis of topology. The best known definition embodying neighborhoods, nearness, and limit ideas is probably that of a topological space. This one is far from satisfactory in many settings and thus other notions appeared (sometimes restricted classes of topological spaces) which attempt to describe some class of topological structures at once small enough to have lots of nice properties and large enough so that all kinds of naturally occurring topological structures in (functional) analysis, algebra, probability, ... would fall under it and such that all kinds of natural constructions (product, spaces of maps, limits, ...) would not take one out of it. For the systematic investigation of this sort of balancing problem, category theory is extremely useful and thus categorical topology arose.

The subject now seems to have reached a certain plateau of maturity, terminology has stabilized and it is definitely time for a first systematic (unifying) textbook on the subject of topological structures, written by one of the active experts in the field. Hence this book: which I hope and expect will be of natural interest to those engaged in research in categorical topology, and will also benefit all those who use topological structures in their work (i.e. almost all mathematicians) but are not necessarily directly active in research in this field itself.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, September 1987

Michiel Hazewinkel

PREFACE

This book is based on lectures the author has given at the Free University Berlin over many years. The first course of this kind took place in summer 1978 in order to prepare my students for attending the International Conference on Categorical Topology (Berlin, August 27-th to September 2-nd, 1978) organized by H. Herrlich and myself. Since Categorical Topology is a fairly young discipline there was no textbook on this subject up to now. The presented course is written for graduate students and interested mathematicians who know already the basic facts on General Topology. Nevertheless some definitions and theorems on uniform spaces are listed up in Chapter 0. Concerning the last chapter of this book the reader is supposed to be acquainted with Algebraic Topology, especially with Čech cohomology theory.

After some preliminary remarks on Set Theory and Category Theory (Chapter 0) topological categories (Chapter 1) are introduced. Then the theory of reflections and coreflections is developped which is also applicable to non-topological categories (Chapter 2). In the following, concrete structures, especially nearness structures, are studied. The interactions of sub- and supercategories of the category Near of nearness spaces (and uniformly continuous maps) are investigated (Chapter 3). Cartesian closedness is studied in Chapter 4 as well as in the more general setting of Chapter 5. Completions are also studied twice, namely for concrete categories as well as for nearness spaces (Chapter 6). Last not least the beautiful relations between dimension theory and cohomology theory known from classical topology are generalized (Chapter 7). In order to be selfcontained representable functors are treated at the end of the book (appendix).

Concerning the presented material not all research areas of Categorical Topology have been included. Especially, I stopped sometimes when the material flowed over to general category theory. Furthermore, I did not treat all applications to other branches of mathematics (e.g. functional analysis or topological algebra); I restricted myself mainly to the field of algebraic topology. Nevertheless I hope that the methods presented will enable the reader to understand all publications on Categorical Topology.

I am very grateful to my friend and colleague Horst Herrlich for his encouragement to publish my lecture notes on Categorical Topology and for his research work that made possible most parts of the book. Further I would like to thank Dipl.-Math. Andreas Behling for translating the main parts of my German manuscript and the Fachbereich Mathematik of the Free University Berlin for paying him. Additionally I thank Mr. Behling for drawing the figures and for preparing the index. I thank Priv.-Doz. Dr. Dr. T. Marny for discussions on the subject presented. Furthermore, I am grateful to Dr. J. Schröder and Dipl.-Math. Olaf Zurth for several parts of the exercises. I thank too Mr. Carsten Scheuch for proofreading. Last not least I thank Mrs. Christa Siewert for her patience in typing the manuscript as well as Mrs. Margrit Barret for assisting her.

Berlin, June 1987

Gerhard Preuß

LIST OF SYMBOLS

Special categories

Ab	244	Pr-Near	111
Bitop	20	PrOrd	146
Born	20	Prox	18
C-Grill	145	PrUConv	150
CGTop	151	PsNear	104
CGUnif	227	PsTop	143
C-Near	108	PsUnif	104
$CompT_2$	50	Reg	22
Cont	108	RegNear	236
Conv	143	$RegNear_1$	238
CReg	22	Rere	21
$CReg_1$	50	R_o-Top	92
$CRegNear_1$	238	S-Conv	145
CSep	86	$SepNear_1$	233
$CSepNear_1$	233	SepUConv	195
Grill	120	Set	5
Haus	67	Simp	21
HConv	194	S-Near	114
HLim	194	SubTop	128
HPsTop	194	Tb-Unif	111
LCon	22	T-Near	92
Lim	18	Top	5
LPCon	22	T-PsTop	144
Meas	20	UConv	149
Mod_R	5	U-Near	97
Near	19	$U\text{-}Near_1$	106
$Near_2$	244	Unif	12
Ord	5	USep	87
P-Near	114		

Notations of some special sets

Let X,Y be sets, $A \subset X$, $B \subset Y$, $f: X \to Y$ a map, $\mathcal{X}$ a topology on X and R an equivalence relation on X .

$\emptyset$ denotes the empty set

$CA := \{x \in X: x \notin A\}$

Sometimes we write $X \setminus A$ instead of CA

$\mathcal{P}(X) := \{A: A \subset X\}$

$f[A] := \{f(x): x \in A\}$

$f^{-1}[B] := \{x \in X: f(x) \in B\}$

$X/_R := \{[x]: [x]$ is an equivalence class with respect to R$\}$

$|X|$ denotes the cardinality of X

$\mathcal{U}_{\mathcal{X}}(x)$ set of neighbourhoods of x with respect to $(X,\mathcal{X})$

$\mathring{\mathcal{U}}_{\mathcal{X}}(x)$ set of open neighbourhoods of x with respect to $(X,\mathcal{X})$

$\bar{A}^{\mathcal{X}}$ closure of A with respect to $(X,\mathcal{X})$

Sometimes we write $\mathcal{U}(x)$, $\mathring{\mathcal{U}}(x)$, $\bar{A}$ instead of $\mathcal{U}_{\mathcal{X}}(x)$, $\mathring{\mathcal{U}}_{\mathcal{X}}(x)$, $\bar{A}^{\mathcal{X}}$.

A^o interior of A with respect to $(X,\mathcal{X})$

$\mathbb{N}$ set of natural numbers

$\mathbb{Z}$ set of integers

$\mathbb{Q}$ set of rational numbers

$\mathbb{R}$ set of real numbers

$(a,b) := \{x \in \mathbb{R}: a < x < b\}$

$(a,b] := \{x \in \mathbb{R}: a < x \leq b\}$

$[a,b) := \{x \in \mathbb{R}: a \leq x < b\}$

$[a,b] := \{x \in \mathbb{R}: a \leq x \leq b\}$

Further symbols

CONTENTS

INTRODUCTION

In order to handle problems of a topological nature, various attempts have been made in the past to introduce suitable concepts, e.g. topological spaces, uniform spaces, proximity spaces, limit spaces, uniform convergence spaces etc. Since this situation was unsatisfactory, new methods were needed to unify all these theories. Thus a new discipline - called *Categorical Topology* - was created (about 1971). It deals with the investigation of topological categories and their relationships to each other. Nevertheless, the search for a suitable 'structure' by means of which any topological concept or idea can be expressed went on. In 1974 H. Herrlich invented nearness spaces, a very fruitful concept which enables one to unify topological and uniform aspects. But to understand the real meaning of this approach a categorical interpretation is useful. Thus decisive parts of this theory belong to Categorical Topology. What are the problems we want to treat in this book?

1) Does there exist a categorical framework for the many kinds of spaces topologists are interested in?
2) What is the categorical background of famous constructions such as the Stone-Čech compactification or the completion of a uniform space?
3) Does there exist some kind of 'structure' which leads to a better approach of topological phenomena than the theory of topological spaces?
4) If the answer to 3) is yes, try to solve the following problems:
   a) Any product of paracompact spaces is paracompact.
   b) Any subspace of a paracompact space is paracompact.
   c) Any subspace of a normal space is normal.
   d) Any product of the unit interval [0,1] with a normal space is normal.
   e) Do the Čech cohomology groups with respect to finite covers fulfill the Eilenberg-Steenrod axioms?

f) Does there exist a cohomological characterization of dimension for topological spaces, uniform spaces and proximity spaces simultaneously?

g) Is it possible to obtain well-known extensions and compactifications of topological spaces by means of the construction of a suitable completion?

5) Find classes of spaces such that the product of two quotient maps is a quotient map and that there is a natural function space structure. Furthermore such a class should not be too 'big' or too 'small' and it should be described by suitable axioms.

6) What is the significance of Dedekind's construction of the real numbers for Categorical Topology?

None of the problems 4) a) - g) can be solved within the framework of topological spaces. But the theory of nearness spaces presented in this book solves the problems. Even 5) finds a satisfactory solution within the realm of 'nearness'.
Problem 1) is solved by the theory of topological categories (resp. initially structured categories), whereas 2) leads to the theory of reflections. Additionally, 5) is directly connected with the theory of cartesian closed topological (resp. initially structured) categories. Last but not least, the further development of Dedekind's construction of the real numbers is the MacNeille completion of a concrete category (problem 6)), whereas the completion of a nearness space (problem 4) g)) generalizes Cantor's construction of the real numbers.

CHAPTER 0

PRELIMINARIES

0.1 Sets, classes and conglomerates

In Cantor's naive set theory every collection of objects specified by some property was called a set. As well-known this approach leads to contradictions, e.g. to the Russell antinomy of the set R of all sets not members of themselves (we obtain

$$[R \in R] \Leftrightarrow [R \notin R]$$

provided R is a set). In order to block this contradiction we introduce two types of collections: classes and sets. Then a class is a collection of objects specified by some property, whereas a set is a class which is a member of some class. Thus R is no set but a (proper) class and also the concept of the class of all sets makes sense. The axiomatic set theory tries to avoid further antinomies. The axiomatic approach of Gödel, Bernays and von Neumann is suitable to handle classes and sets. For further details the interested reader is referred to Dugundji [25] although it is not necessary for understanding this book to know all the details.

Since, occasionally, we will need to consider collections of classes, we introduce the wider concept of conglomerates. Thus, a conglomerate is a collection having classes as members. Especially, we require that conglomerates are closed under the usual set-theoretic constructions (e.g. formation of pairs, unions, products etc.) and that every class is a conglomerate. Therefore conglomerates may be handled like sets. We are allowed to construct functions between them, equivalence relations on them and so on. Some more hints on the axiomatic treatment of the subject can be found in the appendix of the book "Category Theory" by Herrlich and Strecker [44].

0.2 Some categorical concepts

For each mathematical discipline we define at first objects and then admissible maps for describing the objects. This procedure is formalized by the concept 'category'.

0.2.1 Definition. A category $\mathcal{C}$ consists of
(1) a class $|\mathcal{C}|$ of *objects* (which are denoted by $A,B,C,\ldots$),
(2) a class of pairwise disjoint sets $[A,B]_{\mathcal{C}}$ for each pair (A,B) of objects (the members of $[A,B]_{\mathcal{C}}$ are called *morphisms* from A to B), and
(3) a *composition* of morphisms, i.e. for each triple (A,B,C) of objects there is a map

$$[A,B]_{\mathcal{C}} \times [B,C]_{\mathcal{C}} \to [A,C]_{\mathcal{C}}$$

$$(f,g) \mapsto g\circ f$$

(where $\times$ denotes the cartesian product) such that the following axioms are satisfied:

Cat_1) (*Associativity*). If $f \in [A,B]_{\mathcal{C}}$, $g \in [B,C]_{\mathcal{C}}$ and $h \in [C,D]_{\mathcal{C}}$, then $h\circ(g\circ f) = (h\circ g)\circ f$.

Cat_2) (*Existence of identities*). For each $A \in |\mathcal{C}|$ there is an *identity* (*morphism*) $1_A \in [A,A]_{\mathcal{C}}$ such that for all $B,C \in |\mathcal{C}|$, all $f \in [A,B]_{\mathcal{C}}$ and all $g \in [C,A]_{\mathcal{C}}$, $f\circ 1_A = f$ and $1_A\circ g = g$.

0.2.2 Remarks. ① We write $f\colon A \to B$ or $A \xrightarrow{f} B$ instead of $f \in [A,B]_{\mathcal{C}}$. A (resp. B) is called the domain of f (resp. the codomain of f).

② a) The identity 1_A is uniquely determined by A.

b) If $A,A' \in |\mathcal{C}|$ with $A \neq A'$, then $1_A \neq 1_{A'}$, because $[A,A]_{\mathcal{C}} \cap [A',A']_{\mathcal{C}} = \emptyset$.

③ The class of all morphisms of $\mathcal{C}$ is denoted by

$$\text{Mor}\,\mathcal{C} := \bigcup_{(A,B)\in|\mathcal{C}|\times|\mathcal{C}|} [A,B]_{\mathcal{C}} \;;$$

its elements are called C-morphisms.

(4) The requirement of the disjointness of the morphisms sets is not restrictive because it can always be obtained provided that $[A,B]_C$ is replaced by $[A,B]'_C = \{(A,B,\alpha): \alpha \in [A,B]_C\}$.

0.2.3 Examples of categories. (1) The category Set of sets and maps: $|\mathrm{Set}|$ is the class of all sets; $[A,B]_{\mathrm{Set}}$ is the set of all maps from A to B for all $A,B \in |\mathrm{Set}|$. The composition of morphisms is the usual composition of maps.

Remark. In order to obtain the disjointness of the morphisms sets it is useful to define a map $f: A \to B$ as a triple (A,B,F) where $F \subset A \times B$ has the following property: For each $x \in A$ there exists a unique $y \in B$ such that $(x,y) \in F$.

(2) The category Mod_R of R-modules and R-linear maps (where R denotes a commutative ring with unit): $|\mathrm{Mod}_R|$ is the class of all R-modules and Mor Mod_R is the class of all R-linear maps (between any two R-modules). The composition of morphisms is the usual composition of maps.

(3) The category Top of topological spaces (and continuous maps).

(4) The category Ord of ordered sets (and order preserving maps) [An *ordered set* is a pair $(X,\leq)$ where X is a set and $\leq\, \subset X \times X$ is a reflexive, antisymmetric and transitive relation].

(5) If $(S,\leq)$ is an ordered set, then a category C is defined as follows:

$$|C| = S \; ; \; [x,y]_C = \begin{cases} \{(x,y)\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

(6) Let C be a category. Then the dual category C^* is defined as follows:

(1) $|C^*| := |C|$.

(2) $[A,B]_{C^*} := [B,A]_C$ for all $(A,B) \in |C^*| \times |C^*|$.

(3) The composition $\alpha \circ \beta$ in C^* is defined as the composition $\beta \circ \alpha$ in C .