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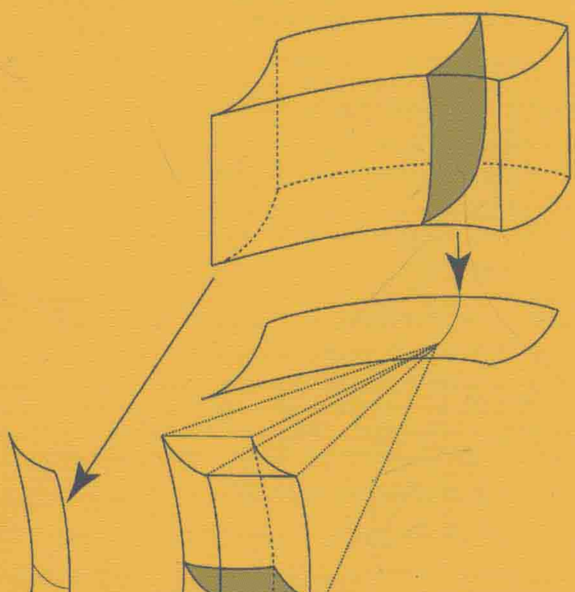

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Preface

This book is about *Symplectic Reduction by Stages* for Hamiltonian systems with symmetry. Reduction by stages means, roughly speaking, that we have two symmetry groups and we want to carry out symplectic reduction by both of these groups, either sequentially or all at once. More precisely, we shall start with a "large group" M that acts on a phase space P and assume that M has a normal subgroup N. The goal is to carry out reduction of the phase space P by the action of M in two stages; first by N and then by the quotient group M/N. For example, M might be the Euclidean group of $\mathbb{R}^3$, with N the translation subgroup so that M/N is the rotation group. In the Poisson context such a reduction by stages is easily carried out and we shall show exactly how this goes in the text. However, in the context of symplectic reduction, things are not nearly as simple because one must also introduce momentum maps and keep track of the level set of the momentum map at which one is reducing. But this gives an initial flavor of the type of problem with which the book is concerned.

As we shall see in this book, carrying out reduction by stages, first by N and then by M/N, rather than all in "one-shot" by the "large group" M is often not only a much simpler procedure, but it also can give non-trivial additional information about the reduced space. Thus, reduction by stages can provide an essential and useful tool for computing reduced spaces, including coadjoint orbits, which is useful to researchers in symplectic geometry and geometric mechanics.

Reduction theory is an old and time-honored subject, going back to the early roots of mechanics through the works of Euler, Lagrange, Poisson, Liouville, Jacobi, Hamilton, Riemann, Routh, Noether, Poincaré, and others. These founding masters regarded reduction theory as a useful tool for simplifying and studying concrete mechanical systems, such as the use of Jacobi's *elimination of the node* in the study of the n-body problem to deal with the overall rotational symmetry of the problem. Likewise, Liouville and Routh used the elimination of cyclic variables (what we would call today an Abelian symmetry group) to simplify problems and it was in this setting that the *Routh stability method* was developed.

The modern form of symplectic reduction theory begins with the works of Arnold [1966a], Smale [1970], Meyer [1973], and Marsden and Weinstein [1974]. A more detailed survey of the history of reduction theory can be found in the first Chapter of the present book. As was the case with Routh, this theory has close connections with the stability theory of *relative equilibria*, as in Arnold [1969] and Simo, Lewis and Marsden [1991]. The symplectic reduction method is, in fact, by now so well known that it is used as a standard tool, often without much mention. It has also entered many textbooks on geometric mechanics and symplectic geometry, such as Abraham and Marsden [1978], Arnold [1989], Guillemin and Sternberg [1984], Libermann and Marle [1987], and McDuff and Salamon [1995]. Despite its relatively old age, research in reduction theory continues vigorously today and this book is a contribution to that theory.

Already in the original papers (such as Marsden and Weinstein [1974]), the issue of performing reduction by stages comes up. That is, one wants a framework in which repeated reduction by two successive symmetry groups can be performed and the result is the same as that of a single larger group. However, even this elementary question has some surprises.

For example, one of the nicest examples of reduction by stages is the theory of *semidirect product reduction* that is due to Guillemin and Sternberg [1980] and Marsden, Ratiu and Weinstein [1984a,b] and which is presented in Chapter 4 of this book. This theory is more than just a verification that reduction for a semidirect product can be done in two stages or, equivalently, all at once. In fact, information and procedures that are useful and powerful in a variety of examples, emerged from that effort. Application areas abound: the heavy top, compressible fluids, magnetohydrodynamics, the dynamics of underwater vehicles, etc. Mathematical techniques, such as the determination of coadjoint orbits in semidirect products, in conjunction with cotangent bundle reduction theory were also developed. Motivated by this success, it was only natural that generalizations would be sought.

In fact, work on a setting for a generalization of semidirect product theory was begun by two of us (JEM and TSR) during a visit to the Schrödinger Institute in Vienna in 1994. After a month or so of thinking about the question, it was realized that while the corresponding question for Poisson reduction was quite simple, the symplectic question was not so easy. It was decided that the framework of starting with a "big group" M with a normal subgroup N and trying to first reduce by N and then by a variant of quotient group M/N was the right framework. Of course the semidirect product case, which was understood at the time, and which is a nontrivial special case, was an important and incentive that provided guidance.

With that modest start, the project slowly evolved and grew in various ways, with a progress report published as Marsden, Misiołek, Perlmutter and Ratiu [1998], and then ending up as this monograph. One of the ways in which it evolved was to ask corresponding questions in the context of Lagrangian reduction. That parallel effort resulted in several important works with Hernan Cendra, the most relevant one for this book being Cendra, Marsden, and Ratiu [2001a] on *Lagrangian Reduction by Stages*. Keeping contact with some of the key applications makes clear the importance of the magnetic terms that appear in the symplectic Hamiltonian and Lagrangian reduction of cotangent and tangent bundles, respectively. Joint work with Darryl Holm on a version of semidirect product reduction theory in the Lagrangian context was an important ingredient (see Holm, Marsden and Ratiu [1998]) in the general Lagrangian reduction by stages program. It was also a key component in the development of the Lagrangian averaged Euler (or Euler-α or LAE) equations as well as the Lagrangian averaged Navier-Stokes equations, also called the LANS-α equations.

Another ingredient that was a driver of some of the initial work was the attempt to understand the relation between cocycles for central extensions (such as the Bott–Virasoro cocycle) and curvatures of connections (such as the mechanical connection) that one uses in the theory of cotangent bundle reduction. These cocycles arise in the study of, for example, the KdV equation and the Camassa-Holm equation (see Ovsienko and Khesin [1987], Misiołek [1997, 1998] and Marsden and Ratiu [1999]) as well as in examples such as spin glasses, as in Holm and Kupershmidt [1988]. We believe that in this book, we have succeeded to a large extent in the interesting task of relating cocycles and magnetic terms. See Cendra, Marsden, and Ratiu [2003] for related ideas.

The work got several important boosts as it proceeded. One was from the PhD thesis of Matt Perlmutter (Perlmutter [1999]) and the second was from a productive visit to Caltech of Gerard Misiołek during 1997–1998. A preliminary version of our results were published in Marsden, Misiołek, Perlmutter and Ratiu [1998], appropriately enough in a celebratory volume for Victor Guillemin. Another boost came in discussions with Juan–Pablo Ortega about how the newly developing theory of optimal reduction based on a distribution-theoretic approach to Hamiltonian conservation laws (Ortega and Ratiu [2002], Ortega [2002]) might fit into the picture. His perspective led to an improvement on and an identification of situations where the hypotheses necessary for reduction by stages (the so called "stages hypothesis") are satisfied. The "optimal" oriented techniques also enabled Juan–Pablo, among other things, to extend the reduction by stages method to the singular case, which appears as Part III of this book.

The singular case is important for many examples; for instance, using the result of Smale [1970] that the momentum map is not regular at points with nontrivial infinitesimal symmetry, Arms, Marsden and Moncrief [1981] showed under rather general circumstances (including in the infinite dimensional case) that the level sets of the momentum map have quadratic singularities. This sort of situation happens in interesting examples, such as Yang-Mills theory and general relativity. There are many other examples of singular reduction, such as those occurring in resonant phenomena (see, for example, Kummer [1981]; Cushman and Rod [1982] and Alber, Luther, Marsden, and Robbins [1998]. It was Sjamaar [1990] and Sjamaar and Lerman [1991] who began the systematic development of the corresponding singular reduction theory. These initial steps, while important, were also limited (they assumed the groups were compact, only dealt with reduction at zero, etc), but the theory rapidly developed in the 1990s and the early 2000s. We summarize some of the key results in this area in §1.4 and will survey additional literature in the historical survey in §1.3. A brief account of singular cotangent bundle reduction is given in §2.4.

Hopefully the above explains how, through this long saga, the three parts of the book came into existence. But it has a happy ending: the theory is not only very attractive, but is now also fairly comprehensive. Of course this does not mean that interesting questions are not left—there are many and we try to point out some of them as we proceed.

Structure of the Book. The book has three parts. The first part gives a fairly complete treatment of regular symplectic reduction, cotangent bundle reduction and also gives an outline of the singular case. We do this for the convenience of the reader as this material is somewhat scattered in the literature. The second part develops the theory of Hamiltonian reduction by stages in the regular case, including a complete treatment of semidirect product reduction theory from the stages point of view. The third part develops this theory in the singular case, that is, the case when the reduced manifolds can have singularities, typically because the symmetry group action is not free, as was mentioned above. While Parts II and III use rather different techniques, the two together make the subject whole. Both theoretically and from the point of view of examples, our view is that it is not helpful to regard the regular case as a special case of the singular case. Thus, we have kept them separate in the two parts.

Prerequisites. It will be assumed that the reader is familiar with the basics of geometric mechanics. While some of this will be recalled as we proceed, this will be mainly for purposes of establishing notation and conventions. In short, we assume everything that is in Marsden and Ratiu [1999], including the construction of momentum maps and their properties; we shall recall, for the reader's convenience, some of the basic theory of symplectic reduction that can be found in, for example, Abraham and Marsden [1978], Marsden [1992] or in one of the many other books on geometric mechanics. As we proceed, we shall review some additional material needed later as well, such as cotangent bundle reduction theory and the theory of principal connections. Again, this is primarily for the reader's convenience.

Part III on singular reduction by stages (by Juan-Pablo Ortega), will require material on singular reduction techniques, so at that point, the reader will need to consult other sources; the main terminology, tools, and techniques are outlined in §1.4 and details can be found in the book Ortega and Ratiu [2004a]. While Part III does not deal with cotangent bundles, an outline of what is known to the authors on singular cotangent bundle reduction is provided in §2.4 for the readers information.

Much of our work on Hamiltonian reduction by stages appears in this book for the first time. The work is just too large to publish in the journal literature without fragmentation and it seemed best to keep it together as a coherent whole.

What is not Covered in this Book. There is a lot that we *do not cover* in this book. As should be clear from the above remarks, we *do focus* on symplectic reduction by stages motivated by both applications and the intrinsic mathematical structure. There are many other aspects of reduction as well, such as Poisson reduction, Lagrangian reduction and Routh reduction. *We do not cover these topics in this work, but a discussion and references are given in the introductory chapter*. For example, Lagrangian reduction itself already deserves a separate monograph, although fairly comprehensive accounts already exist, such as Marsden, Ratiu and Scheurle [2000] and Cendra, Marsden, and Ratiu [2001a].

Another thing we do not cover in this book in a *systematic way* is the analytical (function space) theory in the infinite dimensional case, despite the fact that many of the most interesting examples are, in fact, infinite dimensional. Again this topic deserves a monograph of its own—the *general theory* of infinite dimensional Hamiltonian systems has some way to go, although there has been some general progress, as, for instance, Chernoff and Marsden [1974] and Mielke [1991] and references therein. There are also a number of research papers in this area and we give some specific references in the main text. We give a number of additional comments in §3.2 and, based on Gay-Balmaz and Ratiu [2006] and Gay-Balmaz [2007], outline one example of reduction by stages with all the functional analytic details taken care of in some detail, namely in §9.5 we discuss the case of a fluid in a symmetric container.

We also do not cover the interesting links that reduction theory has with representation theory and quantization and we do not touch all the other interesting developments in symplectic topology. Other than a brief mention in connection with Teichmüller theory in §9.4 and its link to coadjoint orbits of the Bott-Virasoro group, we do not discuss the interesting applications to moduli spaces of connections (see Atiyah and Bott [1982], Goldman and Millson [1990] and Takhtajan and Teo [2004, 2006]).

Apology. As usual in an advanced book with a relatively broad scope, we must apologize in advance to all the researchers in the area whose favorite topic or reference is not found here. Of course it is not possible to be complete with either task as the subject is now too developed and far reaching. However, we would be very happy to receive constructive suggestions for future printings.

Abbreviations. We shall be referring to a few references often, so it will be convenient to have abbreviations for them;

We refer to Abraham and Marsden [1978] as [FofM].

We refer to Abraham, Marsden and Ratiu [1988] as [MTA].

We refer to Marsden and Ratiu [1999] as [MandS].

We refer to Marsden [1992] as [LonM].

We refer to Marsden, Misiołek, Perlmutter and Ratiu [1998] as [MMPR].

We refer to Ortega and Ratiu [2004a] as [HRed].

Notation. To keep things reasonably systematic in the book, we have adopted the following universal conventions for some common maps:

Cotangent bundle projection: $\pi_Q : T^*Q \to Q$

Tangent bundle projection: $\tau_Q : TQ \to Q$

Quotient projection: $\pi_{P,G} : P \to P/G$

Tangent map: $T\varphi : TM \to TN$ for the tangent of a map $\varphi : M \to N$

Thus, for example, the symbol $\pi_{T^*Q,G}$ would denote the quotient projection from T^*Q to $(T^*Q)/G$.

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May, 2007

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Contents

Part I. Background and the Problem Setting

The purpose of this first part is to provide background from those parts of geometric mechanics that are needed in the remainder of the book. The first chapter contains background on regular symplectic reduction and includes all the proofs of the main theorems, such as point reduction, coadjoint orbits and orbit reduction. It also gives an overview of related research topics in geometric mechanics.

The second chapter starts with a review, again including proofs, of connections on principal bundles, including curvature. This is needed background for one of the important constructions for the book, namely cotangent bundle reduction, whose reduced spaces involves non-canonical symplectic structures, namely magnetic, or curvature terms. The chapter also gives the problem setting and explains why reduction by stages is relatively routine in the Poisson setting, while being quite nontrivial in the symplectic setting. The chapter ends with a survey and discussion of various applications, related areas and future directions, such as swimming fish, loop groups, the Bott-Virasoro group, and multisymplectic geometry.

As explained in the preface, the book assumes that the reader is familiar with the essentials of geometric mechanics, up to, but not including symplectic reduction theory; this background is given in, for example, [MandS] (see the abbreviation code for the major references in the preface). Of course people with a strong background in geometric mechanics can proceed directly to Part II.

1

Symplectic Reduction

The purpose of this introductory Chapter is to both establish basic notation and also to give a reasonably complete account of symplectic reduction theory. The first section is a basic introduction, the second provides proofs and the third gives a history of the subject. This chapter focuses on reduction theory in the general setting of symplectic manifolds. The next chapter deals with, amongst other things, the important case of cotangent bundle reduction. Both of these cases are fundamental ingredients in the reduction by stages program.

1.1 Introduction to Symplectic Reduction

Roughly speaking, here is how symplectic reduction goes: given the symplectic action of a Lie group on a symplectic manifold that has a momentum map, one divides a level set of the momentum map by the action of a suitable subgroup to form a new symplectic manifold. Before the division step, one has a manifold (that can be singular if the points in the level set have symmetries) carrying a degenerate closed 2-form. Removing such a degeneracy by passing to a quotient space is a differential-geometric operation that was promoted by Cartan [1922].

The "suitable subgroup" related to a momentum mapping was identified by Smale [1970] in the special context of cotangent bundles. It was Smale's work that inspired the general symplectic construction by Meyer [1973] and the version we shall use, which makes explicit use of the properties of momentum maps, by Marsden and Weinstein [1974].

Momentum Maps. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ be its dual. Suppose that G acts symplectically on a symplectic manifold P with symplectic form denoted by Ω. We shall denote the infinitesimal generator associated with the Lie algebra element ξ by ξ_P and we shall let the Hamiltonian vector field associated to a function $f : P \to \mathbb{R}$ be denoted X_f.

Recall that a ***momentum map*** is a map $\mathbf{J} : P \to \mathfrak{g}^*$, which is defined by the condition

$$\xi_P = X_{\langle \mathbf{J}, \xi \rangle} \tag{1.1.1}$$

for all $\xi \in \mathfrak{g}$, and where $\langle \mathbf{J}, \xi \rangle : P \to \mathbb{R}$ is defined by the natural pointwise pairing. We call such a momentum map ***equivariant*** when it is equivariant with respect to the given action on P and the coadjoint action of G on $\mathfrak{g}^*$. That is,

$$\mathbf{J}(g \cdot z) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(z) \tag{1.1.2}$$

for every $g \in G$, $z \in P$, where $g \cdot z$ denotes the action of g on the point z, Ad denotes the adjoint action, and Ad* the coadjoint action.[1] A quadruple $(P, \Omega, G, \mathbf{J})$, where (P, Ω) is a given symplectic manifold and $\mathbf{J} : P \to \mathfrak{g}^*$ is an equivariant momentum map for the symplectic action of a Lie group G, is sometimes called a ***Hamiltonian G-space***.

Taking the derivative of the equivariance identity (1.1.2) with respect to g at the identity yields the condition of ***infinitesimal equivariance***:

$$T_z \mathbf{J}(\xi_P(z)) = -\,\mathrm{ad}^*_\xi \mathbf{J}(z) \tag{1.1.3}$$

for any $\xi \in \mathfrak{g}$ and $z \in P$. Here, $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}; \eta \mapsto [\xi, \eta]$ is the adjoint map and $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual. A computation shows that (1.1.3) is equivalent to

$$\langle \mathbf{J}, [\xi, \eta] \rangle = \{\langle \mathbf{J}, \xi \rangle, \langle \mathbf{J}, \eta \rangle\} \tag{1.1.4}$$

for any $\xi, \eta \in \mathfrak{g}$, that is, $\langle \mathbf{J}, \cdot \rangle : \mathfrak{g} \to \mathcal{F}(P)$ is a Lie algebra homomorphism, where $\mathcal{F}(P)$ denotes the Poisson algebra of smooth functions on P. The converse is also true if the Lie group is connected, that is, if G is connected then an infinitesimally equivariant action is equivariant (see [MandS], §12.3).

The idea that an action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold P should be accompanied by such an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and the fact that the orbits of this action are themselves symplectic manifolds both occur already in Lie [1890]; the links with

[1] Note that when we write $\mathrm{Ad}^*_{g^{-1}}$, we *literally* mean the adjoint of the linear map $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$. The inverse of g is necessary for this to be a *left* action on $\mathfrak{g}^*$. Some authors let that inverse be understood in the notation. However, in this book, such a convention would be a notational disaster since we need to deal with both *left* and *right* actions, a distinction that is essential in mechanics.