ANALYSIS, MANIFOLDS AND PHYSICS

Part II: 92 Applications

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by

YVONNE CHOQUET-BRUHAT
Membre de l'Académie des Sciences,
Université de Paris VI, Département de Mécanique
Paris, France

CÉCILE DEWITT-MORETTE
University of Texas, Department of Physics

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PREFACE

This book is a companion volume to our first book, *Analysis, Manifolds and Physics* (Revised Edition 1982). In the context of applications of current interest in physics, we develop concepts and theorems, and present topics closely related to those of the first book. The first book is not necessary to the reader interested in Chapters I–V bis and already familiar with differential geometry nor to the reader interested in Chapter VI and already familiar with distribution theory. The first book emphasizes basics; the second, recent applications.

Applications are the lifeblood of concepts and theorems. They answer questions and raise questions. We have used them to provide motivation for concepts and to present new subjects that are still in the developmental stage. We have presented the applications in the forms of problems with solutions in order to stress the questions we wish to answer and the fundamental ideas underlying applications. The reader may also wish to read only the questions and work out for himself the answers, one of the best ways to learn how to use a new tool. Occasionally we had to give a longer-than-usual introduction before presenting the questions. The organization of questions and answers does not follow a rigid scheme but is adapted to each problem.

This book is coordinated with the first one as follows:

1. The chapter headings are the same – but in this book, there is no Chapter VII devoted to infinite dimensional manifolds per se. Instead, the infinite dimensional applications are treated together with the corresponding finite dimensional ones and can be found throughout the book.
2. The subheadings of the first book have not been reproduced in the second one because applications often use properties from several sections of a chapter. They may even, occasionally, use properties from subsequent chapters and have been placed according to their dominant contribution.
3. **Page numbers in parentheses refer to the first book.** References to other problems in the present book are indicated [Problem Chapter Number First Word of Title].

The choice of problems was guided by recent applications of differential geometry to fundamental problems of physics, as well as by our personal interests. It is, in part, arbitrary and limited by time, space, and our desire to bring this project to a close.

The references are not to be construed as an exhaustive bibliography; they are mainly those that we used while we were preparing a problem or that we camë across shortly after its completion.

The book has been enriched by contributions of Charles Doering, Harold Grosse, B. Kent Harrison, N.H. Ibragimov, and Carlos Moreno, and collaborations with Ioannis Bakas, Steven Carlip, Gary Hamrick, Humberto La Roche and Gary Sammelmann. Discussions with S. Blau, M. Dubois-Violette, S.G. Low, L.C. Shepley, R. Stora, A.H. Taub, J. Tits and Jahja Trisnadi are gratefully acknowledged.

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CONVENTIONS

(1) $\{f_n\}_{\mathbb{N}} := \{f_n : n \in \mathbb{N}\}$.

(2) Commutative diagram
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \\ & z \end{tikzcd} \quad \Leftrightarrow \quad \begin{cases} f: x \to y,\ g: y \to z \\ h = g \circ f \end{cases}.$$

(3) Integer part: if $d/2 = 3.5$, then $[d/2] = 3$.

(4) $A \backslash B$ and A/B sometimes mean left and right coset, respectively; but usage varies and is determined in each context.

(5) Exterior product, exterior derivative, interior product

$$(\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\Pi} (\operatorname{sign} \Pi)\Pi[\alpha(v_1, \ldots, v_p) \times \beta(v_{p+1}, \ldots, v_{p+q})],$$

$$(\alpha \bar{\wedge} \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\Pi} (\operatorname{sign} \Pi)\Pi[\alpha(v_1, \ldots, v_p)\beta \times (v_{p+1}, \ldots, v_{p+q})],$$

When operating on a p-form $\bar{d} = d/(p+1)$ and $\bar{i}_v = p i_v$. Note that Kobayashi and Nomizu (Vol. I, p. 35) use what we call $\bar{\wedge}$.

(6) Riemann tensor, Ricci tensor

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\lambda = R_{\alpha\beta}{}^{\lambda}{}_{\mu} v^\mu,$$

i.e.

$$R_{\alpha\beta}{}^{\lambda}{}_{\mu} = \partial_\alpha \Gamma_\beta{}^\lambda{}_\mu - \partial_\beta \Gamma_\alpha{}^\lambda{}_\mu + \Gamma_\alpha{}^\rho{}_\mu \Gamma_\beta{}^\lambda{}_\rho - \Gamma_\beta{}^\rho{}_\mu \Gamma_\alpha{}^\lambda{}_\rho.$$

$$R_{\beta\mu} := R_{\alpha\beta}{}^{\alpha}{}_{\mu}.$$

These conventions agree with Misner, Thorne, and Wheeler and differ from those of our first book *Analysis, Manifolds and Physics.*

(7) A representation of the Dirac matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+,+,+,-)$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Majorana representation of the Dirac matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+,+,+,-)$$

$$\gamma'_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma'_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma'_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \gamma'_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

CONTENTS

I. REVIEW OF FUNDAMENTAL NOTIONS OF ANALYSIS

1. GRADED ALGEBRAS

For applications and references see, for instance, Problems II 1, Super-smooth mappings and III 14, Graded bundles.

A $\mathbb{Z}_2$ **graded algebra** A is a vector space over the field of real or complex numbers which is the direct sum of two subspaces A_+ (called even) and A_- (called odd) — graded algebra

$$A = A_+ \oplus A_-$$

endowed with an associative and distributive operation, called product, such that

$$A_r A_s = A_{r+s}(\text{mod. } 2), \quad r, s = 0, 1, \qquad A_0 = A_+, \qquad A_1 = A_-.$$

A $\mathbb{Z}_2$ graded algebra is called **graded commutative** if any two odd elements anticommute and if even elements commute with all others: — graded commutative

$$ab = (-1)^{d(a)d(b)} ba, \quad a, b \in A$$

where $d(a) = r$ if $a \in A_r$ is the **parity** of a. — parity

We shall consider in this section only graded commutative algebras, so we shall omit the word "commutative".

The algebras we shall use will be endowed with a locally convex Hausdorff topology for which sum and product are continuous operations.

For example, the exterior (Grassmann) algebra over a finite dimensional vector space X (p. 196) is a graded algebra.

A generalization used in physics, which we shall call a (Bryce) **DeWitt algebra** is the algebra B of formal series with a unit e and an infinite number of generators z^I, $I \in \mathbb{N}$, with the usual sum and product laws and the anticommutation property — (Bryce) DeWitt algebra B

$$z^I z^J = -z^J z^I.$$

An element $a \in B$ is written (notion of convergence is irrelevant)

$$a = \sum_{p \in \mathbb{N}} a(p), \qquad a(p) = \frac{1}{p!} a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p}.$$

body **soul** **degree**

$a(0) = a_0 e$ is called the **body** of a, $a_S = \Sigma_{p\geq 1} a(p)$ its **soul.** The numbers a_0, $a_{I_1 \ldots I_p}$ are real or complex, $a_{I_1 \ldots I_p}$ is totally antisymmetric in $I_1 \ldots I_p$; the **degree** of $a(p)$ is p.
B_+ consists of the formal series which contain only terms of even degree, B_- consists of those with only terms of odd degree. B_+ is a subalgebra of B, while B_- is not.

Show that if $ab = 0$ for all $b \in B_+$ [resp. $b \in B_-$] then $a = 0$. Are these properties true in a finitely generated Grassmann algebra?

Answer: If $ab = 0$ for all b belonging to B_+, or to the even part of a finitely generated Grassmann algebra we see that $a = 0$ by taking $b = e$. Suppose now $ab = 0$ for all $b \in B_-$. In particular $az^I = 0$ for each z^I, $I \in \mathbb{N}$. Suppose a coefficient $a_{I_1 \ldots I_p} \neq 0$. Choose $z^J \notin (z^{I_1}, \ldots, z^{I_p})$. We have

$$a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p} z^J \neq 0, \quad \text{hence} \quad a \neq 0.$$

If there is a finite number N of generators the hypothesis $ab = 0$ for all odd b implies only

$$a = cz^1 \ldots z^N, \quad c \text{ arbitrary numbers.}$$ ■

B is endowed with a locally convex, metrizable, Hausdorff topology by the countable family of seminorms (cf. for instance, p. 424)

$$\|a\|_{I_1 \ldots I_p} = |a_{I_1 \ldots I_p}|.$$

The sum of formal series (in particular the decomposition $B = B_+ \oplus B_-$) and their product have the required continuity.

Show that: The partial sums

$$a_m = \sum_{p=0}^{m} a(p)$$

converge to a, in the B-topology, when m tends to infinity.

Answer: If $\| \; \|_{I_1 \ldots I_p}$ is a seminorm on B we have exactly

$$\|a - a_m\|_{I_1 \ldots I_p} = 0 \quad \text{if} \quad m > p.$$

Let $f(x) = \Sigma_{n=0}^{\infty} c_n x^n$ be a numerical series with radius of convergence ρ. Show that $f(a) = \Sigma\, c_n a^n$ is a well-defined formal series in B, depending continuously on a, if $|a_0| < \rho$.

Answer: We have $a = a_0 e + a_S$, so

$$a^n = \sum_{p=0}^{n} C_n^p a_0^{n-p} a_S^p.$$

Since $f(x)$ is convergent for $|x| < \rho$, the numerical series $\Sigma_{n \geq p} c_n C_n^p a_0^{n-p}$ are convergent for $|a_0| < \rho$. We denote their sum by α_p and we write

$$\sum c_n a^n = \sum \alpha_p a_S^p = \sum b(q).$$

Each term on the right-hand side is well defined: $b(q)$ is obtained by finite sums and products since a term of order q arises from a_S^p only when $p \leq q$.
In a similar spirit one proves that the inverse in B of an element a with $a_0 \neq 0$ is the formal series:

$$a^{-1} = a_0^{-1}\left(1 + \sum_n (-1)^n (a_S/a_0)^n\right).$$

2. BEREZINIAN

A **graded matrix** on a graded algebra A is a rectangular array of elements of A, together with a parity attached to each row and column. A square graded matrix with p even and q odd rows and columns is said to be of **order** (p, q). graded matrix

A graded matrix $X = (x^i_j)$ is called **even** [resp. **odd**] if for all i, j: order (p, q) even, odd

$$\mathrm{d}(x^i_j) + \mathrm{d}(i\text{th column}) + \mathrm{d}(j\text{th row}) = 0 \text{ [resp. 1](mod. 2)};$$

one then says that $\mathrm{d}(X) = 0$[resp. $\mathrm{d}(X) = 1$].

1) *We shall always suppose that in a graded matrix X of order (p, q)*

$$X = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$$

the p even rows and columns are written first.
Give the conditions on the parities of the elements of R, S, T, U for X to be even [resp. odd].

Answer: The parities of the columns of R, T are even, those of S, U odd, while the rows of R, S are even and of T, U odd. Thus $\mathrm{d}(X) = 0$ if and only if the elements of R, U are even and the elements of T, S odd. The opposite condition holds for $\mathrm{d}(X) = 1$.

2) *Show that the space $Mat_{p,q}(A)$ of graded matrices of order (p, q) forms a $\mathbb{Z}_2$ graded algebra.*

Answer: The space $\mathrm{Mat}_{p,q}(A)$ obviously forms a vector space over $\mathbb{R}$ or $\mathbb{C}$ (like A), and each element can be written as the sum (usual sum of matrices) of an even and an odd one.
The elements in the product are defined by the usual law

$$XX' = \begin{pmatrix} RR' + ST' & RS' + SU' \\ TR' + UT' & TS' + UU' \end{pmatrix}.$$

It is easy to check that if X and X' have a parity, then

$$\mathrm{d}(XX') = \mathrm{d}(X) + \mathrm{d}(X') (\mathrm{mod.}\ 2).$$

3) *Let B be a DeWitt algebra. Denote by $GL_{p,q}(B)$ the multiplicative group of even invertible graded matrices of order (p, q).*

a) *Let* $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in Mat_{p,q}(B)$, $\mathrm{d}(X) = 0$.
Show that X is invertible if and only if R and U are invertible.

Berezinian

b) The determinant of a square matrix with even elements in B is well defined by the usual polynomial. The **Berezinian** of a matrix $X \in GL_{p,q}(B)$ is the mapping $GL_{p,q}(B) \rightarrow B$ given by

$$\mathrm{Ber}\, X = \det(R - SU^{-1}T)(\det U)^{-1}.$$

Show that Ber X is even valued and invertible.

c) *Show that*

$$\mathrm{Ber}(XY) = \mathrm{Ber}\, X\, \mathrm{Ber}\, Y.$$

Answer a: Under the hypothesis the body of X is

$$X_0 = \begin{pmatrix} R_0 & 0 \\ 0 & U_0 \end{pmatrix}$$

which is invertible if R_0 and U_0 are invertible.

Answer b: Ber X is even because R and U have even elements, S and T odd elements. It is invertible because

$$(\mathrm{Ber}\, X)_0 = (\det R_0)(\det U_0)^{-1} \neq 0.$$

Answer c: The proof is straightforward, in a number of steps (cf. for instance, Leites, p. 16) using in particular the decomposition

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix} = \begin{pmatrix} \mathbb{1}_p & SU^{-1} \\ 0 & \mathbb{1}_q \end{pmatrix} \begin{pmatrix} R - SU^{-1}T & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \mathbb{1}_p & 0 \\ U^{-1}T & \mathbb{1}_q \end{pmatrix}$$

and the fact that any matrix of the form

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

is a product of matrices of the same type, but with a matrix A having only one nonzero element.

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B.S. DeWitt, *Supermanifolds* (Cambridge University Press, London, 1984) and Appendix of "The spacetime approach to quantum field theory", in: *Relativité, Groupes et Topologie II*, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

D.A. Leites, "Introduction to the theory of supermanifolds", Russian Mathematical Surveys **35** (1980) 1.

3. TENSOR PRODUCT OF ALGEBRAS

A real algebra A is a vector space over $\mathbb{R}$ endowed with an associative product, $A \times A \to A$, bilinear with respect to the vector space structure (cf. a more general definition of algebra, p. 9).

1) *Suppose A and B are finite dimensional (as vector spaces) real algebras. Find a natural structure for $A \otimes B$.*

Answer: Let (e_i) and (e_α) be basis for A and B respectively. Then $e_1 \otimes e_\alpha$ is a basis for $A \otimes B$. We define products of such elements by

$$(e_i \otimes e_\alpha)(e_j \otimes e_\beta) = e_i e_j \otimes e_\alpha e_\beta ,$$

where juxtaposition denotes product in the relevant algebra.
The product of arbitrary elements $c = c^{i\alpha} e_i \otimes e_\alpha$, $d = d^{j\beta} e_j \otimes e_\beta$ is given by

$$cd = c^{i\alpha} d^{j\beta} (e_i e_j \otimes e_\alpha e_\beta).$$

It is easy to show that this product has the required properties and is independent of the choice of basis in A and B.

2) *Show that if A is a real algebra, then the complexified algebra $A \otimes \mathbb{C}$ is generated by the complexified vector space $A^{\mathbb{C}}$, that is, the vector space spanned by $a_i e_i$, e_i basis of A, $a^i \in \mathbb{C}$.*

Answer: A basis of $\mathbb{C}$ as a real vector space is $(1, \mathrm{i})$, and the algebra structure is determined by $\mathrm{i}^2 = -1$; a basis of $A \otimes \mathbb{C}$ is $(e_j \otimes 1, e_j \otimes \mathrm{i})$, which we can denote $(e_j, \mathrm{i}e_j)$ without breaking the product law.

3) *Example*: *Tensor products of matrices (see Problem I 4, Clifford algebras). Let A be the space of $n \times n$ matrices and B be the space of $m \times m$ matrices. Construct $a \otimes b$ for $a \in A$, $b \in B$.*

Answer: Let $a = (a^i_j)$, $b = (b^\alpha_\beta)$ be respectively an $n \times n$ and an $m \times m$ matrix. Then $a \otimes b = ((a \otimes b)^I_J)$, where the indices I and J stand for a pair of indices (i, α) or (j, β) and $(a \otimes b)^I_J = a^i_j b^\alpha_\beta$. Usually one orders pairs of indices as follows: $(1,1), (1,2), \ldots, (2,1), (2,2), \ldots$.

Note: In Problem IV 2, Obstruction, following Atiyah, Bott and Shapiro, we shall use the graded tensor product of two graded algebras defined as follows. Let $A = \Sigma_{i=0,1} A^i$ and $B = \Sigma_{i=0,1} B^i$ be two graded algebras. The **graded tensor product** $A \hat{\otimes} B$ is, by definition, the algebra whose underlying vector space is $\Sigma_{i,j=0,1} A^i \otimes B^j$ with multiplication defined by

graded tensor product

$$(u \otimes x_i)(y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v,$$

where x_i [resp. y_j] is an element of B^i [resp. A^j], u[resp. v] is an arbitrary element of A [resp. B].

The graded tensor product is again a graded algebra

$$(A \hat{\otimes} B)^k = \sum A^i \otimes B^j, \qquad i + j = k \bmod 2.$$

For example, consider the odd element $e_1 \otimes 1 + 1 \otimes e_2$, its square $e_1^2 \otimes 1 + 1 \otimes e_2^2$ is even.

4. CLIFFORD ALGEBRAS

1. INTRODUCTION

Let V be a real $d = n + m$ dimensional vector space with a pseudo-euclidean scalar product g, invariant under the group $\mathrm{O}(n, m)$, given by $g = (g_{AB})$, $g_{AB} = 0$ if $A \neq B$, $g_{AA} = 1$, $A = 1, \ldots, n$, $g_{AA} = -1$ if $A = n+1, \ldots, n+m$. The Clifford algebra (p. 65) $\mathscr{C}(n, m)$ is the real vector space endowed with an associative product, distributive with respect to