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AMS Subject Classifications (1980): 31B10, 31B20, 35J05, 35J25, 45B05, 45P05

ISBN 3-540-10227-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-10227-2 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Král, Josef, DrSc. Integral operators in potential theory. (Lecture notes in mathematics; 823) Bibliography: p. Includes indexes. 1. Potential, Theory of. 2. Integral operators. I. Title. II. Series: Lecture notes in mathematics (Berlin); 823.
QA3.L28. no. 823. [QA404.7]. 510s. [515.7] 80-23501

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

CONTENTS

Introductory remark

We shall be concerned with relations of analytic properties of classical potential theoretic operators to the geometry of the corresponding domain in the Euclidean m-space R^m , $m \geqq 2$.

Let us recall that a function h is termed harmonic in an open set $G \subset R^m$ if it is twice continuously differentiable in G and satisfies there the so-called Laplace equation

$$\Delta h \equiv \sum_{i=1}^{m} \partial_i^2 h = 0 ,$$

where ∂_i denotes the partial derivative with respect to the i-th variable. (In fact, such a function h is necessarily infinitely differentiable and even real-analytic; this is usually proved in elementary theory of harmonic functions on account of the Poisson integral which will be derived in the example following theorem 2.19.) If we try to determine a harmonic function in $R^m \setminus \{0\}$ of the form $h(x) = \omega(|x|)$, where ω is an unknown function with a continuous second derivative in $]0,+\infty[\subset R^1$, we obtain an ordinary differential equation

$$\frac{d^2\omega(r)}{dr^2} + \frac{m-1}{r}\frac{d\omega(r)}{dr} = 0$$

whose solutions are

$$\omega(r) = \begin{cases} \alpha\, r^{2-m} + \beta & \text{in case} \quad m > 2 \,, \\ \alpha \log r + \beta & \text{in case} \quad m = 2 \,, \end{cases}$$

where α, β are arbitrary constants.

Let us denote by $A \equiv A_m$ the area of the unit sphere in R^m, i.e.

$$A_m = \frac{2\,\pi^{\frac{1}{2}m}}{\Gamma(\frac{1}{2}\,m)}$$

where $\Gamma(.)$ is the Euler gamma-function, and put for $x \in \in R^m \setminus \{0\}$

$$h_o(x) = \begin{cases} \frac{1}{(m-2)A}\,|x|^{2-m} & \text{if} \quad m > 2 \,, \\ \frac{1}{A} \log \frac{1}{|x|} & \text{if} \quad m = 2 \,. \end{cases}$$

As we shall observe later, with this normalization - h_o represents a fundamental solution of the Laplace equation which means that, in the sense of distribution theory,

$$-\Delta h_o = \delta_o \,,$$

where δ_o is the Dirac measure (= unit point mass situated at the origin). For $m = 3$ the function h_o occurred first in physics; according to the Newton gravitation law (or Coulomb's law in electrostatics) the vector-valued function

$$x \longmapsto \operatorname{grad} h_o(x)$$

describes the force-field of a point mass (or a point charge) placed at the origin.

The function h_z defined by

$$h_z(x) = \begin{cases} h_o(x-z) & \text{for } x \neq z\ , \\ +\infty & \text{for } x = z\ , \end{cases}$$

is sometimes called the fundamental harmonic function with pole at z . It follows from the above remark that h_z is, up to additive and multiplicative constants, the only harmonic function in $R^m \setminus \{z\}$ whose values depend on the distance from z only. If $G \subset R^m$ is open, then all the functions h_z with $z \in R^m \setminus G$ as well as their directional derivatives

$$x \longmapsto n \cdot \text{grad}_x\, h_z(x)$$

(where $n \in R^m$) are harmonic in G . The idea of using "combinations" of these simple functions for generating more complicated harmonic functions in G is classical. By "combinations" here we mean not only discrete combinations but in general the integrals of the form

$$\mathcal{U}\nu(x) = \int_{R^m} h_z(x) d\nu(z)\ , \tag{1}$$

$$W\nu(x) = \int_{R^m} n(z)\cdot \text{grad}_x\, h_z(x) d\nu(z)\ , \tag{2}$$

where ν is a signed measure with support in $R^m \setminus G$ and $z \longmapsto n(z)$ is a suitable vector-valued function with values in R^m . In classical potential theory G is usually supposed to have a smooth boundary B with area element ds and ν is taken in the form $d\nu = fds$, where f is an appropriate function on B , while $n(z)$ is usually chosen as the unit normal to B . The integrals of the form (2) are then called the double layer potentials and proved to be useful in connection with the Dirichlet problem which reads as follows: Given a continuous function g on B , determine a harmonic function h in G such that $\lim_{\substack{x \to z \\ x \in G}} h(x) = g(z)$ for every $z \in B$. If h is taken in the form of a double layer potential (2) with the above described specification for n and $d\nu = f ds$, then evaluation of the limit at $z \in B$ leads to an integral equation of the second kind

$$f(z) + \int_B K(z,y)f(y)ds(y) = 2g(z)$$

for the unknown density f . In a similar way, the so-called single layer potentials (1) are useful in treating the Neumann problem which is formulated as follows: Given a function g on B , determine a harmonic function h in G such that $\lim_{\substack{x \to z \\ x \in G}} n(z) \cdot \operatorname{grad} h(x) = g(z)$ for every $z \in B$, where $n(z)$ is the unit exterior normal to G at z . If one tries $h = \mathcal{U}\nu$ with $d\nu = f ds$, this problem again reduces to an integral equation of the second kind for the unknown density f and the kernel of the corresponding integral operator is transposed to the kernel resulting from the Dirichlet problem for the complementary domain. Historically it was this method of treating boundary value problems in potential theory that led to the development of the Fredholm theory of equations of the second kind. In its classical formulation the method is tied up with certain à priori smoothness restrictions on the boundary of the domain, because the normal derivative occurs in the definition of double layer potentials and in the formulation of the Neumann problem. These restrictions may be entirely avoided, however, if the normal derivative is characterized weakly. Normal derivatives of single layer potentials as well as double layer potentials may then be introduced and investigated for general open sets $G \subset R^m$ without any à priori restrictions on the boundary. Some results in this direction together with their applications to boundary value problems will be described below.

§ 1

Weak normal derivatives of potentials

We shall denote by $\mathcal{D} \equiv \mathcal{D}(R^m)$ the class of all infinitely differentiable functions with compact support in R^m.

1.1. Definition. Let h be a harmonic function in an open set $G \subset R^m$ and suppose that

$$\int_P |\operatorname{grad} h(x)|\,dx < \infty$$

for every bounded open set $P \subset G$. Then $Nh \equiv N^G h$ will denote the functional over $\mathcal{D}$ defined by

$$\langle \varphi, Nh \rangle = \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h(x)\,dx , \qquad \varphi \in \mathcal{D} .$$

Nh will be termed the generalized normal derivative of h.

Remark. The reason for this terminology lies in the fact that, in the case when G is bounded by a smooth closed surface B with area element ds and exterior normal $n = (n_1,\dots,n_m)$ and when the partial derivatives $\partial_i h$ $(i = 1,\dots,m)$ extend from G to continuous functions on the whole $G \cup B$, the Gauss-Green formula yields

$$\langle \varphi, Nh \rangle = \int_B \varphi\Big(\sum_{i=1}^{m} n_i\, \partial_i h\Big)ds , \qquad \varphi \in \mathcal{D} .$$

Consequently, Nh is a natural weak characterization of the normal derivative $\sum_{i=1}^{m} n_i \, \partial_i h = \frac{\partial h}{\partial n}$.

1.2. Remark. If G and h have the meaning described in the definition 1.1, then $\langle \varphi, N^G h \rangle = 0$ for every $\varphi \in \in \mathcal{D}$ whose support does not meet the boundary of G. In other words, the support of $N^G h$ is contained in the boundary of G.

Proof. Suppose that the support of $\varphi \in \mathcal{D}$ does not meet the boundary of G and define $\tilde{\varphi}$ so that $\tilde{\varphi} = \varphi$ in G, $\tilde{\varphi} = 0$ on $R^m \setminus G$. Clearly, $\tilde{\varphi} \in \mathcal{D}$ and if $\tilde{h}$ is any twice continuously differentiable function on R^m coinciding with h near the support of $\tilde{\varphi}$, then

$$\int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h(x) dx = \int_{R^m} \operatorname{grad} \tilde{\varphi}(x) \cdot \operatorname{grad} \tilde{h}(x) dx =$$

$$= -\int_{R^m} \tilde{\varphi}(x) \Delta \tilde{h}(x) dx = 0 ,$$

because $\tilde{\varphi} \Delta \tilde{h} = 0$ everywhere.

1.3. Notation. The ball of radius r and center y in R^m will be denoted by

$$\Omega(r,y) \equiv \Omega_r(y) = \{x \in R^m;\ |x-y| < r\}.$$

For $M \subset R^m$ we denote by diam M the diameter of M, by cl M the closure of M, and by $\mathcal{H}_k(M)$ the (outer) k-dimensional Hausdorff measure of M. Let us recall that

$$\mathcal{H}_k(M) = \lim_{\varepsilon \to 0+} \mathcal{H}_k^{\varepsilon}(M) ,$$

where

$$\mathcal{H}_k^{\varepsilon}(M) = 2^{-k} V_k \inf \sum_n (\operatorname{diam} M_n)^k$$

with the infimum taken over all sequences of sets $M_n \subset R^m$ such that $\operatorname{diam} M_n \leqq \varepsilon$ and $\bigcup_n M_n \supset M$ and with

$$V_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$$

equal to the volume of the unit ball in k-space. The normalization is chosen in such a way that $\mathcal{H}_m(M)$ coincides with the outer m-dimensional Lebesgue measure of M ; if M is a simple smooth k-dimensional surface in R^m , then $\mathcal{H}_k(M)$ coincides with the area of M . (Basic facts concerning Hausdorff measures may be found in the monograph [Ro] .)

By a signed measure we mean a finite σ-additive set function defined on the σ-algebra of Borel sets in R^m . If ν is a signed measure and $M \subset R^m$ is a Borel set, then $|\nu|(M)$ denotes the total variation of ν on M ; we put $\|\nu\| = |\nu|(R^m)$. If $B \subset R^m$ is compact, we denote by $\mathcal{E}'(B)$ the linear space of all signed measures ν with $|\nu|(R^m \setminus B) = 0$, i.e. with support in B ; $\mathcal{E}'(B)$ is a Banach space if equipped with the norm $\|\ldots\|$. The abbreviation spt ν (spt $\varphi, \ldots$) will denote the support of ν (support of $\varphi, \ldots$). If $\nu \in \mathcal{E}'(B)$, then the potential $\mathcal{U}\nu(x)$ defined by (1) is meaningful for all $x \in R^m \setminus B$ and represents a harmonic function in $R^m \setminus B$. The symbol δ_y will denote the Dirac measure defined by

$$\delta_y(M) = \begin{cases} 1 & \text{if } y \in M , \\ 0 & \text{if } y \notin M \end{cases}$$

on Borel sets $M \subset R^m$. Thus $\mathcal{U}\delta_y = h_y$ on R^m . We put $\Gamma = \{x \in R^m;\ |x| = 1\}$, so that $A = \mathcal{H}_{m-1}(\Gamma)$.

1.4. Remark. The following elementary transformation formula will be often useful below:

If g is an integrable function on R^m and $z \in R^m$ is fixed, then the function

$$\theta \longmapsto \int_0^\infty t^{m-1} g(z+t\,\theta)\,dt$$

is defined for $\mathcal{H}_{m-1}$-almost every $\theta \in \Gamma$, is integrable $d\mathcal{H}_{m-1}$ and

$$\int_{R^m} g(x)\,dx = \int_\Gamma \Big(\int_0^\infty t^{m-1} g(z+t\,\theta)\,dt\Big)\,d\,\mathcal{H}_{m-1}(\theta) .$$

Remark. If $B \subset R^m$ is compact and $\nu \in \mathcal{C}'(B)$, then for $x \in R^m \setminus B$

$$|\operatorname{grad} \mathcal{U}\nu(x)| \leqq \frac{1}{A}\int_B |x-z|^{1-m}\,d|\nu|(z) ,$$

whence we get for any bounded Borel set $P \subset R^m \setminus B$

$$\int_P |\operatorname{grad} \mathcal{U}\nu(x)|\,dx \leqq \frac{1}{A}\int_B \Big(\int_P |x-z|^{1-m}\,dx\Big)\,d\,|\nu|(z) \leqq$$

$$\leqq \operatorname{diam}(P \cup B)\|\nu\| < \infty .$$

We see that if $G \subset R^m$ is an open set with a compact boundary B, then $N^G \mathcal{U}\nu$ (taken in the sense of the definition 1.1) is available for every $\nu \in \mathcal{C}'(B)$.

Example. Fix $z \in R^m$ and let $G = R^m \setminus \{z\}$, $B = \{z\}$, $\nu = \delta_z$. Employing the transformation formula in 1.4 one gets easily for $\varphi \in \mathcal{D}$

$$\langle \varphi, N^G \mathcal{U}\delta_z \rangle = \int_{R^m \setminus \{z\}} \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h_z(x)dx =$$

$$= \frac{1}{A} \int_{R^m} |z-x|^{1-m} \operatorname{grad} \varphi(x) \cdot \frac{z-x}{|z-x|}\, dx = \varphi(z) .$$

We see that $N^G \mathcal{U}\delta_z = \delta_z$ in this case.

Noting that

$$\int_{R^m} \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h_z(x)dx = - \int_{R^m} \Delta\varphi(x)h_z(x)dx$$

we may rewrite the above equality in the form

$$\int_{R^m} \Delta\varphi(x)h_z(x)dx = -\varphi(z), \quad \varphi \in \mathcal{D} .$$

This means that, in the language of distribution theory, $\Delta h_z = -\delta_z$.

1.5. Observation. If $G \subset R^m$ is an open set with a compact boundary B and $\nu \in \mathcal{C}'(B)$ then, for any $\varphi \in \mathcal{D}$,

(3) $\langle \varphi, N^G \mathcal{U}\nu \rangle = \int_B \langle \varphi, N^G \mathcal{U}\,\delta_y \rangle d\nu(y)$.

Proof. Fix $\varphi \in \mathcal{D}$ and put $c = \sup|\operatorname{grad}\varphi(x)|$, $P = G \cap \operatorname{spt}\varphi$. Elementary calculation yields the estimate

$$(4_1) \quad \iint_{G \times B} |\operatorname{grad}\varphi(x) \cdot \operatorname{grad} h_y(x)|\, dx\, d|\nu|(y) \leqq$$

$$\leqq c \operatorname{diam}(P \cup B) \|\nu\|$$

which shows that the double integral

$$(4_2) \quad \iint_{G \times B} \operatorname{grad}\varphi(x) \operatorname{grad} h_y(x)\, dx\, d\nu(y)$$

converges. It remains to apply Fubini's theorem and note that the two repeated integrals derived from (4_2) occur in (3).

1.6. Some questions. Let $G \subset R^m$ be an open set with a compact boundary B . For every $\nu \in \mathcal{E}'(B)$ we have then the generalized normal derivative $N^G \mathcal{U}\nu$ of the corresponding potential defined as a functional over $\mathcal{D}$. If there is a signed measure μ such that

$$\langle \varphi, N^G \mathcal{U}\nu \rangle = \int_{R^m} \varphi\, d\mu , \qquad \varphi \in \mathcal{D} ,$$

then we shall say, as usual, that $N^G \mathcal{U}\nu$ is a measure and write $N^G \mathcal{U}\nu = \mu$; in this case necessarily $\mu \in \mathcal{E}'(B)$ by remark 1.2. In general, however, $N^G \mathcal{U}\nu$ need not be a measure. We thus arrive naturally at the following

Question 1. Under which conditions on G can we assert that $N^G \mathcal{U} \nu \in \mathcal{C}'(B)$ for every $\nu \in \mathcal{C}'(B)$?

Our main objective in this paragraph is to answer this question in geometric terms connected with G . Before doing so we shall investigate the following simplified problem.

Question 2. Let $y \in B$ be a fixed point. What geometric conditions on G guarantee that $N^G \mathcal{U} \delta_y \in \mathcal{C}'(B)$?

In order to be able to answer this question we first introduce suitable terminology and establish several auxiliary results.

1.7. Definition. Let $S, M \subset R^m$. A point $y \in S$ will be termed a hit of S on M if for every $r > 0$ both

$$\mathcal{H}_1(\Omega_r(y) \cap S \cap M) > 0 \quad \text{and} \quad \mathcal{H}_1(\Omega_r(y) \cap (S \setminus M)) > 0 .$$

(In our applications S usually will be a straight line segment or a half-line.)

1.8. Lemma. Let $M \subset R^1$ be a Borel set and denote by χ_M its characteristic function on R^1 . If $a < b$, then

$$(5) \quad \sup\left\{ \int_a^b \chi_M(t)\psi'(t)dt \ ; \quad \psi \in \mathcal{D} \ , \quad |\psi| \leqq 1, \quad \operatorname{spt} \psi \subset \,]a,b[\right\}$$

equals the total number of hits of $]a,b[$ on M (which is $+\infty$ if the set of all hits of $]a,b[$ on M is infinite).

<u>Proof.</u> Let q be the number of all hits of $]a,b[$ on M. Suppose first that $q < +\infty$ and let $a_1 < \dots < a_q$ be all the hits of $]a,b[$ on M. Then no $]a_j, a_{j+1}[$ can meet both M and $R^1 \setminus M$ in a set of positive linear measure. It follows that either M or $]a,b[\setminus M$ is $\mathcal{H}_1$-equivalent with

$$\bigcup_k \,]a_{2k-1}, a_{2k}[\;, \quad \text{where} \quad 1 \leqq k \;, \quad 2k \leqq q \;.$$

If $\psi \in \mathcal{D}$ and $\operatorname{spt} \psi \subset]a,b[$, then

$$\int_a^b \varkappa_M(t)\, \psi'(t)\, dt = \pm \sum_{j=1}^{q} (-1)^j \psi(a_j)$$

and the supremum (5) equals q.

Next suppose that the supremum (5) is finite. This means that the functional

$$L : \psi \longmapsto \int_a^b \varkappa_M(t)\, \psi'(t)\, dt$$

is bounded on the space $\mathcal{D}(]a,b[)$ of all infinitely differentiable functions ψ with $\operatorname{spt} \psi \subset]a,b[$ with respect to the norm $\|\psi\| = \sup_t |\psi(t)|$. Referring to the Hahn--Banach extension theorem and Riesz representation theorem we conclude that there is a function g of bounded variation on $]a,b[$ such that

$$\langle \psi, L \rangle = \int_a^b \psi \, dg = - \int_a^b g(t)\, \psi'(t)\, dt \;, \quad \psi \in \mathcal{D}(]a,b[) \,.$$