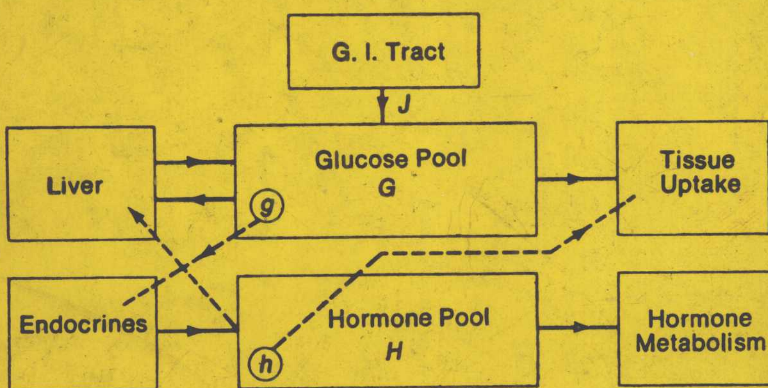
Martin Braun

Differential Equations and Their Applications

Short Version



$$\frac{dg}{dt} = -m_1 g - m_2 h + J(t)$$

$$\frac{dh}{dt} = -m_3 h + m_4 g$$

Springer International Student Edition

Martin Braun

Differential Equations and Their Applications

Short Version

Springer-Verlag Berlin Heidelberg New York
1978

Martin Braun
Department of Mathematics, Queens College, City University of New York, Flushing, N.Y. 11367, USA

SPRINGER INTERNATIONAL STUDENT EDITION

ISBN 3-540-78017-3

Authorised reprint of the original edition published by Springer-Verlag New York Inc. This book is an abridged version of Differential Equations and Their Applications by Martin Braun, © 1975, 1978 by Springer-Verlag New York Inc.

ISBN 0-387-90289-9 Springer-Verlag New York
0-540-90289-9 Springer-Verlag Berlin Heidelberg

Preface

This textbook is a unique blend of the theory of differential equations and their exciting application to "real world" problems. First, and foremost, it is a rigorous study of ordinary differential equations and can be fully understood by anyone who has completed one year of calculus. However, in addition to the traditional applications, it also contains many exciting "real life" problems. These applications are completely self contained. First, the problem to be solved is outlined clearly, and one or more differential equations are derived as a model for this problem. These equations are then solved, and the results are compared with real world data. The following applications are covered in this text.

1. In Section 1.3 we prove that the beautiful painting "Disciples at Emmaus" which was bought by the Rembrandt Society of Belgium for $170,000 was a modern forgery.

2. In Section 1.5 we derive differential equations which govern the population growth of various species, and compare the results predicted by our models with the known values of the populations.

3. In Section 1.6 we try to determine whether tightly sealed drums filled with concentrated waste material will crack upon impact with the ocean floor. In this section we also describe several tricks for obtaining information about solutions of a differential equation that cannot be solved explicitly.

4. In Section 2.7 we derive a very simple model of the blood glucose regulatory system and obtain a fairly reliable criterion for the diagnosis of diabetes.

5. In Section 4.3 we derive two Lanchestrian combat models, and fit one of these models, with astonishing accuracy, to the battle of Iwo Jima in World War II.

This textbook also contains the following important, and often unique features.

1. In Section 1.9 we give a complete proof of the existence–uniqueness theorem for solutions of first-order equations. Our proof is based on the method of Picard iterates, and can be fully understood by anyone who has completed one year of calculus.

2. Modesty aside, Section 2.12 contains an absolutely super and unique treatment of the Dirac delta function. We are very proud of this section because it eliminates all the ambiguities which are inherent in the traditional exposition of this topic.

3. All the linear algebra pertinent to the study of systems of equations is presented in Sections 3.1–3.5. One advantage of our approach is that the reader gets a concrete feeling for the very important but extremely abstract properties of linear independence, spanning, and dimension. Indeed, many linear algebra students sit in on our course to find out what's really going on in their course.

I greatly appreciate the help of the following people in the preparation of this manuscript: Eleanor Addison who drew the original figures, and Kate MacDougall, Sandra Spinacci, and Miriam Green who typed portions of this manuscript.

I am grateful to Walter Kaufmann-Bühler, the mathematics editor at Springer-Verlag, and Elizabeth Kaplan, the production editor, for their extensive assistance and courtesy during the preparation of this manuscript. It is a pleasure to work with these true professionals.

Finally, I am especially grateful to Joseph P. LaSalle for the encouragement and help he gave me. Thanks again, Joe.

New York City
October, 1977

Martin Braun

Contents

First-order differential equations 1

1.1 Introduction

This book is a study of differential equations and their applications. A differential equation is a relationship between a function of time and its derivatives. The equations

$$\frac{dy}{dt} = 3y^2 \sin(t+y) \qquad \text{(i)}$$

and

$$\frac{d^3y}{dt^3} = e^{-y} + t + \frac{d^2y}{dt^2} \qquad \text{(ii)}$$

are both examples of differential equations. The order of a differential equation is the order of the highest derivative of the function *y* that appears in the equation. Thus (i) is a first-order differential equation and (ii) is a third-order differential equation. By a solution of a differential equation we will mean a continuous function *y(t)* which together with its derivatives satisfies the relationship. For example, the function

$$y(t) = 2\sin t - \tfrac{1}{3}\cos 2t$$

is a solution of the second-order differential equation

$$\frac{d^2y}{dt^2} + y = \cos 2t$$

since

$$\frac{d^2}{dt^2}\left(2\sin t - \tfrac{1}{3}\cos 2t\right) + \left(2\sin t - \tfrac{1}{3}\cos 2t\right)$$
$$= \left(-2\sin t + \tfrac{4}{3}\cos 2t\right) + 2\sin t - \tfrac{1}{3}\cos 2t = \cos 2t.$$

Differential equations appear naturally in many areas of science and the humanities. In this book, we will present serious discussions of the applications of differential equations to such diverse and fascinating problems as the detection of art forgeries, the diagnosis of diabetes, the growth of cancerous tumor cells, the battle of Iwo Jima during World War II, and the growth of various populations. Our purpose is to show how researchers have used differential equations to solve, or try to solve, *real life* problems. And while we will discuss some of the great success stories of differential equations, we will also point out their limitations and document some of their failures.

1.2 First-order linear differential equations

We begin by studying first-order differential equations and we will assume that our equation is, or can be put, in the form

$$\frac{dy}{dt} = f(t,y). \tag{1}$$

The problem before us is this: Given $f(t,y)$ find all functions $y(t)$ which satisfy the differential equation (1). We approach this problem in the following manner. A fundamental principle of mathematics is that the way to solve a new problem is to reduce it, in some manner, to a problem that we have already solved. In practice this usually entails successively simplifying the problem until it resembles one we have already solved. Since we are presently in the business of solving differential equations, it is advisable for us to take inventory and list all the differential equations we can solve. If we assume that our mathematical background consists of just elementary calculus then the very sad fact is that the only first-order differential equation we can solve at present is

$$\frac{dy}{dt} = g(t) \tag{2}$$

where g is any integrable function of time. To solve Equation (2) simply integrate both sides with respect to t, which yields

$$y(t) = \int g(t)\,dt + c.$$

Here c is an arbitrary constant of integration, and by $\int g(t)\,dt$ we mean an anti-derivative of g, that is, a function whose derivative is g. Thus, to solve any other differential equation we must somehow reduce it to the form (2). As we will see in Section 1.8, this is impossible to do in most cases. Hence, we will not be able, without the aid of a computer, to solve most differential equations. It stands to reason, therefore, that to find those differential equations that we *can* solve, we should start with very simple equations and not ones like

$$\frac{dy}{dt} = e^{\sin(t-37\sqrt{|y|})}$$

(which incidentally, cannot be solved exactly). Experience has taught us that the "simplest" equations are those which are *linear* in the dependent variable y.

Definition. The general first-order linear differential equation is

$$\frac{dy}{dt} + a(t)y = b(t). \tag{3}$$

Unless otherwise stated, the functions $a(t)$ and $b(t)$ are assumed to be continuous functions of time. We single out this equation and call it linear because the dependent variable y appears by itself, that is, no terms such as e^{-y}, y^3 or $\sin y$ etc. appear in the equation. For example $dy/dt = y^2 + \sin t$ and $dy/dt = \cos y + t$ are both *nonlinear* equations because of the y^2 and $\cos y$ terms respectively.

Now it is not immediately apparent how to solve Equation (3). Thus, we simplify it even further by setting $b(t) = 0$.

Definition. The equation

$$\frac{dy}{dt} + a(t)y = 0 \tag{4}$$

is called the ***homogeneous*** first-order linear differential equation, and Equation (3) is called the ***nonhomogeneous*** first-order linear differential equation for $b(t)$ not identically zero.

Fortunately, the homogeneous equation (4) can be solved quite easily. First, divide both sides of the equation by y and rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t).$$

Second, observe that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt}\ln|y(t)|$$

where by $\ln|y(t)|$ we mean the natural logarithm of $|y(t)|$. Hence Equation (4) can be written in the form

$$\frac{d}{dt}\ln|y(t)| = -a(t). \tag{5}$$

But this is Equation (2) "essentially" since we can integrate both sides of (5) to obtain that

$$\ln|y(t)| = -\int a(t)\,dt + c_1$$

where c_1 is an arbitrary constant of integration. Taking exponentials of both sides yields

$$|y(t)| = \exp\left(-\int a(t)\,dt + c_1\right) = c\exp\left(-\int a(t)\,dt\right)$$

or

$$\left|y(t)\exp\left(\int a(t)\,dt\right)\right| = c. \tag{6}$$

Now, $y(t)\exp\left(\int a(t)\,dt\right)$ is a continuous function of time and Equation (6) states that its absolute value is constant. But if the absolute value of a continuous function $g(t)$ is constant then g itself must be constant. To prove this observe that if g is not constant, then there exist two different times t_1 and t_2 for which $g(t_1)=c$ and $g(t_2)=-c$. By the intermediate value theorem of calculus g must achieve all values between $-c$ and $+c$ which is impossible if $|g(t)|=c$. Hence, we obtain the equation $y(t)\exp\left(\int a(t)\,dt\right)=c$ or

$$y(t) = c\exp\left(-\int a(t)\,dt\right). \tag{7}$$

Equation (7) is said to be the *general solution* of the homogeneous equation since every solution of (4) must be of this form. Observe that an arbitrary constant c appears in (7). This should not be too surprising. Indeed, we will always expect an arbitrary constant to appear in the general solution of any first-order differential equation. To wit, if we are given dy/dt and we want to recover $y(t)$, then we must perform an integration, and this, of necessity, yields an arbitrary constant. Observe also that Equation (4) has infinitely many solutions; for each value of c we obtain a distinct solution $y(t)$.

Example 1. Find the general solution of the equation $(dy/dt)+2ty=0$.

Solution. Here $a(t)=2t$ so that $y(t)=c\exp\left(-\int 2t\,dt\right)=ce^{-t^2}$.

Example 2. Determine the behavior, as $t\to\infty$, of all solutions of the equation $(dy/dt)+ay=0$, a constant.

Solution. The general solution is $y(t)=c\exp\left(-\int a\,dt\right)=ce^{-at}$. Hence if $a<0$, all solutions, with the exception of $y=0$, approach infinity, and if $a>0$, all solutions approach zero as $t\to\infty$.

In applications, we are usually not interested in all solutions of (4). Rather, we are looking for the *specific* solution $y(t)$ which at some initial time t_0 has the value y_0. Thus, we want to determine a function $y(t)$ such that

$$\frac{dy}{dt}+a(t)y=0, \qquad y(t_0)=y_0. \tag{8}$$

Equation (8) is referred to as an initial-value problem for the obvious reason that of the totality of all solutions of the differential equation, we are looking for the one solution which initially (at time t_0) has the value y_0. To find this solution we integrate both sides of (5) between t_0 and t. Thus

$$\int_{t_0}^{t}\frac{d}{ds}\ln|y(s)|\,ds=-\int_{t_0}^{t}a(s)\,ds$$

and, therefore

$$\ln|y(t)|-\ln|y(t_0)|=\ln\left|\frac{y(t)}{y(t_0)}\right|=-\int_{t_0}^{t}a(s)\,ds.$$

Taking exponentials of both sides of this equation we obtain that

$$\left|\frac{y(t)}{y(t_0)}\right|=\exp\left(-\int_{t_0}^{t}a(s)\,ds\right)$$

or

$$\left|\frac{y(t)}{y(t_0)}\exp\left(\int_{t_0}^{t}a(s)\,ds\right)\right|=1.$$

The function inside the absolute value sign is a continuous function of time. Thus, by the argument given previously, it is either identically $+1$ or identically -1. To determine which one it is, evaluate it at the point t_0; since

$$\frac{y(t_0)}{y(t_0)}\exp\left(\int_{t_0}^{t_0}a(s)\,ds\right)=1$$

we see that

$$\frac{y(t)}{y(t_0)}\exp\left(\int_{t_0}^{t}a(s)\,ds\right)=1.$$

Hence

$$y(t)=y(t_0)\exp\left(-\int_{t_0}^{t}a(s)\,ds\right)=y_0\exp\left(-\int_{t_0}^{t}a(s)\,ds\right).$$

Example 3. Find the solution of the initial-value problem

$$\frac{dy}{dt}+(\sin t)y=0, \qquad y(0)=\tfrac{3}{2}.$$

Solution. Here $a(t)=\sin t$ so that

$$y(t)=\tfrac{3}{2}\exp\left(-\int_0^t \sin s\, ds\right)=\tfrac{3}{2}e^{(\cos t)-1}.$$

Example 4. Find the solution of the initial-value problem

$$\frac{dy}{dt}+e^{t^2}y=0, \qquad y(1)=2.$$

Solution. Here $a(t)=e^{t^2}$ so that

$$y(t)=2\exp\left(-\int_1^t e^{s^2}ds\right).$$

Now, at first glance this problem would seem to present a very serious difficulty in that we cannot integrate the function e^{s^2} directly. However, this solution is equally as valid and equally as useful as the solution to Example 3. The reason for this is twofold. First, there are very simple numerical schemes to evaluate the above integral to any degree of accuracy with the aid of a computer. Second, even though the solution to Example 3 is given explicitly, we still cannot evaluate it at any time t without the aid of a table of trigonometric functions and some sort of calculating aid, such as a slide rule, electronic calculator or digital computer.

We return now to the nonhomogeneous equation

$$\frac{dy}{dt}+a(t)y=b(t).$$

It should be clear from our analysis of the homogeneous equation that the way to solve the nonhomogeneous equation is to express it in the form

$$\frac{d}{dt}(\text{“something”})=b(t)$$

and then to integrate both sides to solve for “something”. However, the expression $(dy/dt)+a(t)y$ does not appear to be the derivative of some simple expression. The next logical step in our analysis therefore should be the following: Can we make the left hand side of the equation to be d/dt of “something”? More precisely, we can multiply both sides of (3) by any continuous function $\mu(t)$ to obtain the equivalent equation

$$\mu(t)\frac{dy}{dt}+a(t)\mu(t)y=\mu(t)b(t). \tag{9}$$

(By equivalent equations we mean that every solution of (9) is a solution of (3) and vice-versa.) Thus, can we *choose* $\mu(t)$ so that $\mu(t)(dy/dt)+a(t)\mu(t)y$ is the derivative of some simple expression? The answer to this question is yes, and is obtained by observing that

$$\frac{d}{dt}\mu(t)y=\mu(t)\frac{dy}{dt}+\frac{d\mu}{dt}y.$$

Hence, $\mu(t)(dy/dt)+a(t)\mu(t)y$ will be equal to the derivative of $\mu(t)y$ if and only if $d\mu(t)/dt=a(t)\mu(t)$. But this is a first-order linear homogeneous equation for $\mu(t)$, i.e. $(d\mu/dt)-a(t)\mu=0$ which we already know how to solve, and since we only need one such function $\mu(t)$ we set the constant c in (7) equal to one and take

$$\mu(t)=\exp\left(\int a(t)\,dt\right).$$

For this $\mu(t)$, Equation (9) can be written as

$$\frac{d}{dt}\mu(t)y=\mu(t)b(t). \tag{10}$$

To obtain the general solution of the nonhomogeneous equation (3), that is, to find all solutions of the nonhomogeneous equation, we take the indefinite integral (anti-derivative) of both sides of (10) which yields

$$\mu(t)y=\int \mu(t)b(t)\,dt+c$$

or

$$y=\frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt+c\right)=\exp\left(-\int a(t)\,dt\right)\left(\int \mu(t)b(t)\,dt+c\right). \tag{11}$$

Alternately, if we are interested in the specific solution of (3) satisfying the initial condition $y(t_0)=y_0$, that is, if we want to solve the initial-value problem

$$\frac{dy}{dt}+a(t)y=b(t), \qquad y(t_0)=y_0$$

then we can take the definite integral of both sides of (10) between t_0 and t to obtain that

$$\mu(t)y-\mu(t_0)y_0=\int_{t_0}^{t}\mu(s)b(s)\,ds$$

or

$$y=\frac{1}{\mu(t)}\left(\mu(t_0)y_0+\int_{t_0}^{t}\mu(s)b(s)\,ds\right). \tag{12}$$

Remark 1. Notice how we used our knowledge of the solution of the homogeneous equation to find the function $\mu(t)$ which enables us to solve the nonhomogeneous equation. This is an excellent illustration of how we use our knowledge of the solution of a simpler problem to solve a harder problem.

Remark 2. The function $\mu(t)=\exp\left(\int a(t)\,dt\right)$ is called an *integrating factor* for the nonhomogeneous equation since after multiplying both sides by $\mu(t)$ we can immediately integrate the equation to find all solutions.

Remark 3. The reader should not memorize formulae (11) and (12). Rather, we will solve all nonhomogeneous equations by first multiplying both sides by $\mu(t)$, by writing the new left-hand side as the derivative of $\mu(t)y(t)$, and then by integrating both sides of the equation.

Remark 4. An alternative way of solving the initial-value problem $(dy/dt)+a(t)y=b(t)$, $y(t_0)=y_0$ is to find the general solution (11) of (3) and then use the initial condition $y(t_0)=y_0$ to evaluate the constant c. If the function $\mu(t)b(t)$ cannot be integrated directly, though, then we must take the definite integral of (10) to obtain (12), and this equation is then approximated numerically.

Example 5. Find the general solution of the equation $(dy/dt)-2ty=t$.
Solution. Here $a(t)=-2t$ so that

$$\mu(t)=\exp\left(\int a(t)\,dt\right)=\exp\left(-\int 2t\,dt\right)=e^{-t^2}.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain the equivalent equation

$$e^{-t^2}\left(\frac{dy}{dt}-2ty\right)=te^{-t^2}\quad\text{or}\quad\frac{d}{dt}e^{-t^2}y=te^{-t^2}.$$

Hence,

$$e^{-t^2}y=\int te^{-t^2}dt+c=\frac{-e^{-t^2}}{2}+c$$

and

$$y(t)=-\tfrac{1}{2}+ce^{t^2}.$$

Example 6. Find the solution of the initial-value problem

$$\frac{dy}{dt}+2ty=t,\qquad y(1)=2.$$

Solution. Here $a(t)=2t$ so that

$$\mu(t)=\exp\left(\int a(t)\,dt\right)=\exp\left(\int 2t\,dt\right)=e^{t^2}.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain that

$$e^{t^2}\left(\frac{dy}{dt}+2ty\right)=te^{t^2}\quad\text{or}\quad\frac{d}{dt}(e^{t^2}y)=te^{t^2}.$$

Hence,

$$\int_1^t \frac{d}{ds} e^{s^2} y(s)\,ds = \int_1^t s e^{s^2}\,ds$$

so that

$$e^{s^2} y(s)\Big|_1^t = \frac{e^{s^2}}{2}\Big|_1^t.$$

Consequently,

$$e^{t^2} y - 2e = \frac{e^{t^2}}{2} - \frac{e}{2}$$

and

$$y = \frac{1}{2} + \frac{3e}{2} e^{-t^2} = \tfrac{1}{2} + \tfrac{3}{2} e^{1-t^2}.$$

Example 7. Find the solution of the initial-value problem

$$\frac{dy}{dt} + y = \frac{1}{1+t^2}, \quad y(2) = 3.$$

Solution. Here $a(t) = 1$, so that

$$\mu(t) = \exp\left(\int a(t)\,dt\right) = \exp\left(\int 1\,dt\right) = e^t.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain that

$$e^t\left(\frac{dy}{dt} + y\right) = \frac{e^t}{1+t^2} \quad \text{or} \quad \frac{d}{dt} e^t y = \frac{e^t}{1+t^2}.$$

Hence

$$\int_2^t \frac{d}{ds} e^s y(s)\,ds = \int_0^t \frac{e^s}{1+s^2}\,ds,$$

so that

$$e^t y - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\,ds$$

and

$$y = e^{-t}\left[3e^2 + \int_2^t \frac{e^s}{1+s^2}\,ds\right].$$

Exercises

In each of Problems 1–7 find the general solution of the given differential equation.

1. $\dfrac{dy}{dt} + y\cos t = 0$

2. $\dfrac{dy}{dt} + y\sqrt{t}\,\sin t = 0$

3. $\frac{dy}{dt}+\frac{2t}{1+t^2}y=\frac{1}{1+t^2}$

4. $\frac{dy}{dt}+y=te^t$

5. $\frac{dy}{dt}+t^2y=1$

6. $\frac{dy}{dt}+t^2y=t^2$

7. $\frac{dy}{dt}+\frac{t}{1+t^2}y=1-\frac{t^3}{1+t^4}y$

In each of Problems 8–14, find the solution of the given initial-value problem.

8. $\frac{dy}{dt}+\sqrt{1+t^2}\,y=0, \qquad y(0)=\sqrt{5}$

9. $\frac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \qquad y(0)=1$

10. $\frac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \qquad y(0)=0$

11. $\frac{dy}{dt}-2ty=t, \qquad y(0)=1$

12. $\frac{dy}{dt}+ty=1+t, \qquad y(\tfrac{3}{2})=0$

13. $\frac{dy}{dt}+y=\frac{1}{1+t^2}, \qquad y(1)=2$

14. $\frac{dy}{dt}-2ty=1, \qquad y(0)=1$

15. Find the general solution of the equation

$$(1+t^2)\frac{dy}{dt}+ty=(1+t^2)^{5/2}.$$

(*Hint*: Divide both sides of the equation by $1+t^2$.)

16. Find the solution of the initial-value problem

$$(1+t^2)\frac{dy}{dt}+4ty=t, \qquad y(1)=\tfrac{1}{4}.$$

17. Find a continuous solution of the initial-value problem

$$y'+y=g(t), \qquad y(0)=0$$

where

$$g(t)=\begin{cases} 2, & 0\leqslant t\leqslant 1 \\ 0, & t>1 \end{cases}.$$

18. Show that every solution of the equation $(dy/dt)+ay=be^{-ct}$ where a and c are positive constants and b is any real number approaches zero as t approaches infinity.

19. Given the differential equation $(dy/dt)+a(t)y=f(t)$ with $a(t)$ and $f(t)$ continuous for $-\infty<t<\infty$, $a(t)\geqslant c>0$, and $\lim_{t\to\infty} f(t)=0$, show that every solution tends to zero as t approaches infinity.

When we derived the solution of the nonhomogeneous equation we tacitly assumed that the functions $a(t)$ and $b(t)$ were continuous so that we could perform the necessary integrations. If either of these functions was discontinuous at a point t_1, then we would expect that our solutions might be discontinuous at $t=t_1$. Problems 20–23 illustrate the variety of things that