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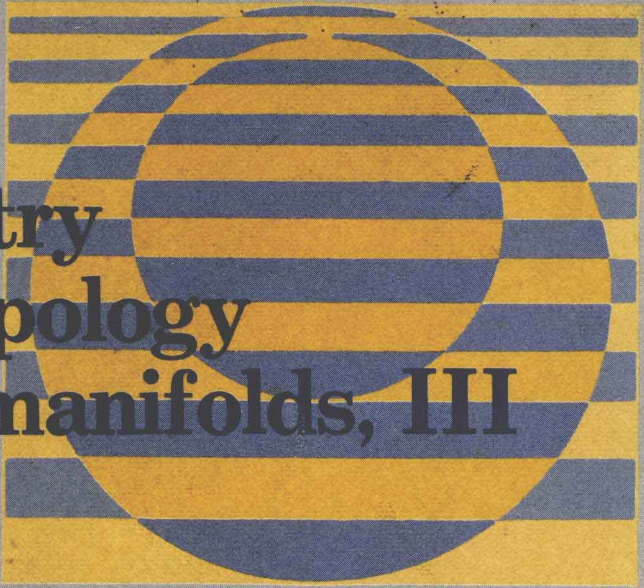

# Geometry and Topology of Submanifolds, III

14–18 May 1990

*Editors*

**Leopold Verstraelen**
**Alan West**

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**GEOMETRY AND TOPOLOGY OF SUBMANIFOLDS, III**



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# PREFACE

This volume contains the proceedings of the international meeting "Workshop on surfaces, submanifolds and their applications" organised at Leeds, England, from May 14 till May 18, 1990. We do believe that this meeting turned out to be rather successfull: it offered many really beautiful lectures on fundamental recent progress in the field of differential geometry of submanifolds, and throughout the meeting there was a lively exchange of ideas and information between all participants.

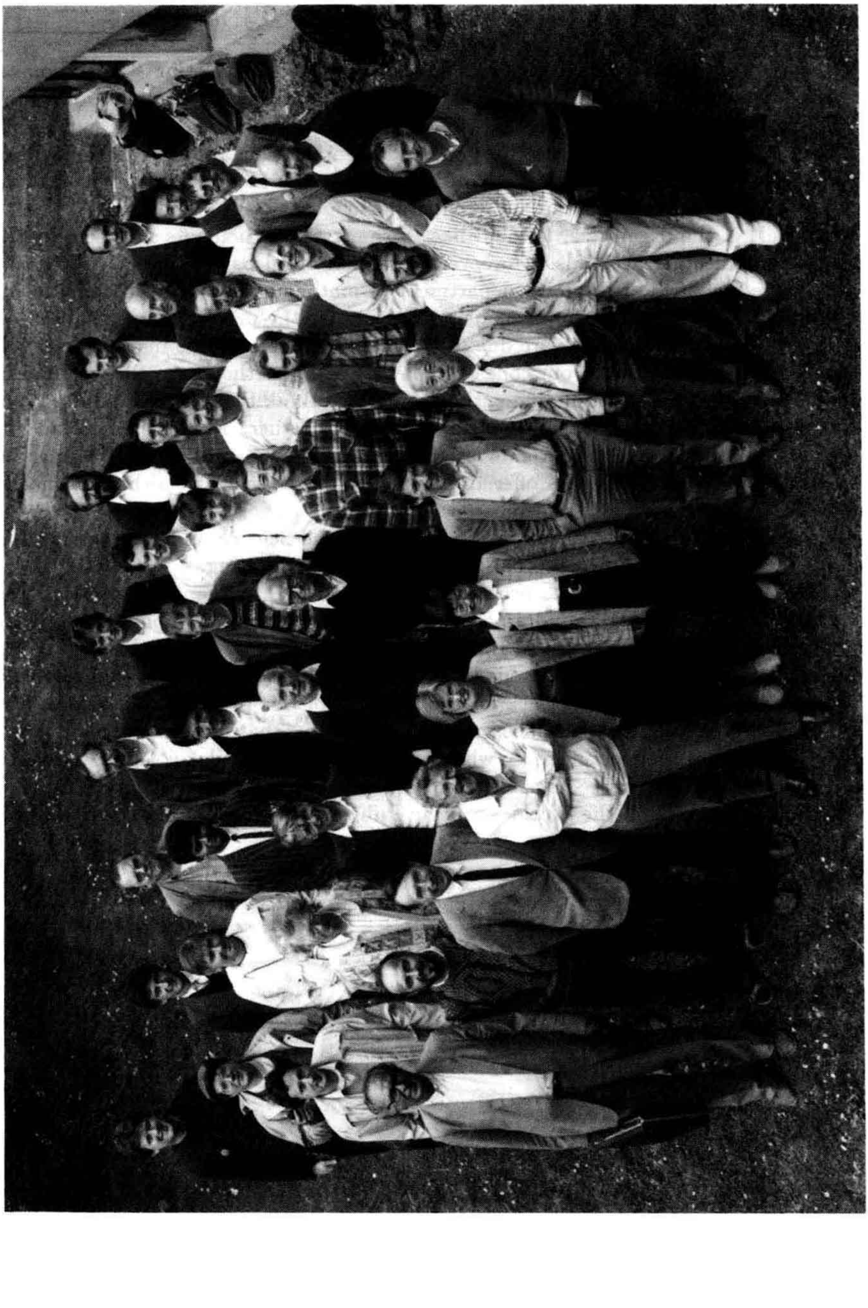
On behalf of the organisers, I would like to thank all participants for their contribution to our meeting at Leeds. Moreover, on behalf of all participants, I would like to thank the members of the local department of mathematics, both mathematicians and administrative personel, for their help in caring for the practical organisation. In particular, many special thanks are due to the main organiser of this meeting, our colleague Alan West, and also to our colleague Sheila Carter; as they form the well known Leeds-"tandem" for research in differential geometry, I think that they also did a lot of the work related to the organisation of this meeting together. Moreover, I would like to express our thanks to Mrs. Alan West for her work done during this meeting. In short, mainly these three people offered us a week at Leeds which, both scientifically and socially, will be very well remembered indeed !

We thank the London Mathematical Society and the Science and Engineering Research Council for their generous financial support to our meeting.

Concerning the editing of the proceedings, I would like to thank Alan West, and also my coworker Georges Zafindratafa at Leuven, for their help. Some authors are more slowly than others to send in their paper. Based on my own experience, I can understand that there may be reasons to ask for some postponement of the originally planned deadline, and then later ask for the iteration of this process... As before, also now I have been accepting this, since, mainly, I do believe that a more complete list of papers presented at the meeting, even when published with some delay, is to be prefered over rather incomplete proceedings even appearing very shortly after the meeting.

As usual, we do thank the World Scientific Publishing Co for publishing these proceedings.

Leopold Verstraelen



# WORKSHOP PHOTOGRAPH

FRONT ROW (Left to right)

Ebrahin Esraphilian, Josef Dorfmeister, Patrick Ryan, Alan West, Sheila Carter, Chuu-Lian Terng, Brian Smyth, Katsumi Nomizu, Michael Clancy, John Burns.

SECOND ROW

Stephen Slebarski, Hermann Karcher, Peter Jupp, Arthur Ledger, James Eells, Richard Palais, Gudlaugur Thorbergsson, Steve Buyske, Helmut Reckziegel.

THIRD ROW

Ulrich Pinkall, Douglas Clark, Vijay Parmar, Ridvan Ezentas, Luc Vrancken, Barbara Opozda, Jean-Marie Morvan, Fransisco Urbano, John Wood.

FOURTH ROW

Ignace Van de Woestijne, Martin Magid, Stewart Robertson, Kinetsu Abe.

FIFTH ROW

Sigmundur Gudmunsson, Kadri Arslan, Francis Burstall, Tom Wheldon, Franki Dillen, Leopold Verstraelen, Ryszard Deszcz, Thomas Cecil.

# Geometry and Topology of Submanifolds, III

# CONTENTS

# AFFINE DIFFERENTIAL GEOMETRY OF COMPLEX HYPERSURFACES

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## 1 Introduction

In this paper a complex analogue of affine differential geometry of real hypersurfaces is discussed.

Let $M^n$ be a holomorphic hypersurface of $\mathbf{A}^{n+1}$ , which is $\mathbf{C}^{n+1}$ regarded as an affine space. The group of complex affine transformations, then, consists of the complex special linear group and the translations of $\mathbf{C}^{n+1}$.

In the real case [2], it is of fundamental importance that there exists a unique transversal vector field called the affine normal field of $M^n$ .

This paper begins with establishing a set of n+2 transversal (local) vector fields which play the role of the affine normal field. Any one of these vector fields can be obtained from another by multiplying an (n+2)-nd root of unity. They are, as a set, invariant under the affine transformations which leave $M^n$ fixed as a whole.

The technique used here can, perhaps, be called the method of complex moving frames. The papers of Chern [4] and Flanders [5] have been especially helpful. Indeed, the first half of this paper may be regarded as a complex modification and refinement of theirs.

The second half deals with the determination of the homogeneous complex affine surfaces in $\mathbf{A}^3$. A homogeneous complex affine surface in $\mathbf{A}^3$ is a surface which is the orbit of a point in $\mathbf{A}^3$ under a subgroup action of the affine transformation group.

The result states that, except for the quadrics, there are basically only several nondegenerate homogeneous complex affine surfaces in $\mathbf{A}^3$ , see Theorems 2 and 3.

The scheme of the proof is similar to that of Guggenheimer [6]. Also see Jensen [8]. That surfaces are holomorphic, in fact, makes the argument less complicated than the real case.

As a concluding remark, a set of counterexamples for the complex version of Jörgens' theorem [9] is presented to show that the analogy to the real case often breaks down. The reasons are often obvious, although sometimes less clear.

In a forthcoming paper [1], a more comprehensive study of the affine differential geometry of complex hypersurfaces will be taken up. There some affine connections in the hypersurfaces will be introduced and more geometric aspects will be emphasized. It is therefore hoped that similarity and difference of the real and complex affine differential geometries will more lucidly emerge.

Finally, many thanks are due to Professor K. Nomizu. Through his lectures in the seminars at Brown, the author was introduced to the affine differential geometry of real hypersurfaces.

The author also thanks M. Magid and E. Spiegel for the useful conversations with them during the preparation of this paper.

## 2 Complex Affine Hypersurfaces

Let $\mathbf{A}^{n+1}$ denote the complex $(n+1)$-dimensional Euclidean space with the flat (affine) connection. By the (complex) special affine transformation group $\mathbf{SA}(n+1)$ of $\mathbf{A}^{n+1}$ , we mean the natural semi-direct product of $\mathbf{SL}(n+1, \mathbf{C})$ and $\mathbf{C}^{n+1}$ . Here $\mathbf{SL}(n+1, \mathbf{C})$ denotes the set of all $(n+1) \times (n+1)$-complex matrices with determinant 1, and $\mathbf{C}^{n+1}$ represents the group of translations in $\mathbf{A}^{n+1}$ under the addition in $\mathbf{C}^{n+1}$ . $\mathbf{SA}(n+1)$ has the following well known matrix representation, with respect to which the group operation becomes the usual matrix product:

$$\begin{bmatrix} & a_1 \\ & \cdot \\ A & \cdot \\ & \cdot \\ & a_{n+1} \\ 0\cdots 0 & 1 \end{bmatrix}, \quad A \in \mathbf{SL}(n+1,\mathbf{C}) \ \textit{and} \ \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n+1} \end{bmatrix} \in \mathbf{C}^{n+1}. \tag{1}$$

In particular, an element of $\mathbf{SL}(n+1,\mathbf{C})$ will be often referred to as an unimodular transformation.

Let $\chi : M^{n+1} \to \mathbf{A}$ be a holomorphic immersion of a complex n-dimensional manifold $M^n$ into $\mathbf{A}^{n+1}$ with the standard complex structure. We will denote by $TM^n$ the complex tangent bundle of $M^n$, i.e., the holomorphic part of the complexified tangent bundle. In like manner, we will represent by $TM^n_x$ the complex tangent space of $M^n$ at $x$.

By a (complex) frame $(x, e_1, \ldots, e_{n+1})$ of a point $x$ in $\mathbf{A}^{n+1}$, one means a pair of a point $x$ in $\mathbf{A}^{n+1}$ and $(n+1)$ (complex) vectors $e_1, \ldots, e_n$ in $\mathbf{C}^{n+1}$ and such that $e_1, \ldots, e_{n+1}$ are obtained from the standard cartesian coordinate vectors $(1, \ldots, 0), \ldots, (0, \ldots, 1)$ in $\mathbf{C}^{n+1}$ via an element of $\mathbf{SL}(n+1,\mathbf{C})$.

As a convention, given (n+1) vectors $X_1, \cdots, X_{n+1}$ in $\mathbf{C}^{n+1}$, $|X_1, \cdots, X_{n+1}|$ will mean the determinant of the $(n+1)\times(n+1)$-matrix obtained from $X_1, \cdots, X_{n+1}$ by regarding them as column vectors of the matrix in the natural sense. Hence, for a frame $(x, e_1, \cdots, e_{n+1})$, $|e_1, \cdots, e_{n+1}| = 1$.

From now on (for the time being, anyway), we will be interested only in holomorphic frames; namely, the frame represented by holomorphic cross-sections of the frame bundle. A frame thus will mean a holomorphic frame unless otherwise specified. We also employ the convention that all the Greek letters $\alpha, \beta, \gamma$ etc. run 1 through $n+1$ and all the English letters $a, b, c$ etc. run 1 through $n$. We also adopt the well accepted convention in tensor calculus; whenever the same letter appears as a superscript and a subscript, the summation over the letter must be performed.

Now considering the holomorphic immersion $\chi$ as a vector-valued function, one may take its exterior derivative $d\chi$ , resulting a holomorphic vector valued 1-form. We will denote it by :

$$d\chi = \omega^{\alpha} e_{\alpha}, \tag{2}$$

where $\omega^{\alpha}$ is a holomorphic 1-form.

For each $e_{\alpha}$ , we also have

$$de_{\alpha} = \omega_{\alpha}^{\beta} e_{\beta} \tag{3}$$

Here $\omega_{\alpha}^{\beta}$ is a holomorphic 1-form.

Integrability condition gives rise to the well known structural equations:

$$d\omega^{\alpha} = \omega^{\beta} \wedge \omega_{\beta}^{\alpha}. \tag{4}$$

$$d\omega_{\beta}^{\alpha} = \omega_{\beta}^{\gamma} \wedge \omega_{\gamma}^{\alpha}. \tag{5}$$

Since

$$0 = d1 = d \mid e_1, \cdots, e_{n+1} \mid = \sum_{\alpha} \mid e_1, \cdots, de_{\alpha}, \cdots, e_{n+1} \mid,$$

one has

$$\omega_{\alpha}^{\alpha} = 0. \tag{6}$$

If one chooses the frames whose first n vectors are tangential to $M^n$ , we get $\omega^{n+1} = 0$ on $M^n$ . Hence,

$$d\omega^{n+1} = \omega^{\beta} \wedge \omega_{\beta}^{n+1} = \omega^{i} \wedge \omega_{i}^{n+1} = 0. \tag{7}$$

Applying Cartan's lemma to (7), one gets holomorphic functions $h_{ij}$ such that

$$\omega_{i}^{n+1} = h_{ij}\omega^{j} \quad \text{and} \quad h_{ij} = h_{ji}. \tag{8}$$

Thus, given a frame, one gets a (complex) symmetric quadratic form

$$\amalg = \omega^{i}\omega_{i}^{n+1} = h_{ij}\omega^{i}\omega^{j}.$$

Call $\amalg$ the second fundamental form of $\chi$ relative to the frame. Notice that $\amalg$ is a holomorphic quadratic form.

Let $(x, e_1^*, \cdots, e_{n+1}^*)$ be another holomorphic frame associated with $\chi$, i.e., $e_1^*, \cdots, e_{n+1}^*$ are tangential to $\chi$ at $x \in M^n$ . Then there is a field of holomorphic unimodular transformations $\mathbf{A} = (a_{\beta}^{\alpha})$ such that

$$e_i^* = a_i^k e_k \quad \text{and} \quad e_{n+1}^* = a^{-1}e_{n+1} + a_{n+1}^{i}e_i. \tag{9}$$

Here $a_{n+1}^{n+1} = a^{-1} = \left[\det(a_i^j)\right]^{-1}$ *and* $\mathrm{a}_i^{n+1} = 0$ .

(9) can be represented in a matrix product:

$$[e_\alpha^*] = \begin{bmatrix} & & 0 \\ & & \cdot \\ & A & \cdot \\ & & \cdot \\ & & 0 \\ a_{n+1}^1, & \cdots, a_{n+1}^n & a^{-1} \end{bmatrix} [e_\alpha] \ , \ A = (a_j^i). \tag{10}$$

Reversing the role of $[e_\alpha]$ and $[e^*_\alpha]$ we get an $(n+1)\times(n+1)$ -matrix $\mathbf{B}=(b^\beta_\alpha)$ such that

$$\begin{bmatrix} & & 0 \\ & & \cdot \\ & B & \cdot \\ & & \cdot \\ & & 0 \\ b^1_{n+1}, & \cdots, b^n_{n+1} & a \end{bmatrix}, \tag{11}$$

where, $B=(b^i_k)=A^{-1}$, or $a^j_i b^i_k=\delta^j_k$ and $\det B=a^{-1}$.

The structural equations (2) and (3) for $(x,e^*)$ become

$$d\chi={\omega^*}^i e^*_i \quad \text{and} \quad de^*_\alpha={\omega^*}^\beta_\alpha e^*_\beta \tag{12}$$

There exist the following relations between $[\omega^\alpha]$ and $[\omega^{*\alpha}]$ , corresponding to (10) and (11) ; denoting by $\mathbf{A}^T$ and $\mathbf{B}^T$ the transposes of $\mathbf{A}$ and $\mathbf{B}$, respectively,

$$[\omega^\alpha]=\mathbf{A}^T[\omega^{*\alpha}] \quad \text{and} \quad [\omega^{*\alpha}]=\mathbf{B}^T[\omega^\alpha]. \tag{13}$$

The forms $\omega^{n+1}_i$ and ${\omega^*}^{n+1}_i$ are related as followed:

$$\begin{aligned} \omega^{n+1}_i &= |\, e_1,\cdots,e_n, de_i \,| \\ &= |\, b^k_i e^*_k,\cdots,b^k_n e^*_k, d(b^k_i e^*_k) \,| \\ &= |\, b^k_l e^*_k,\cdots,b^k_n e^*_k, b^k_i {\omega^*}^{n+1}_k e^*_{n+1} \,| \\ &= a^{-1} b^k_i {\omega^*}^{n+1}_k. \end{aligned} \tag{14}$$

Thus, the second fundamental forms $\coprod$ and $\coprod^*$ relative to the frames $(\chi, e_\alpha)$ and $(\chi, e^*_\alpha)$ are related as follows. From (8) and (14),

$$\coprod = \omega^i \omega_i^{n+1} = a^i_j \omega^{*j} a^{-1} b^k_i \omega^{*n+1}_k = \delta^k_j a^{-1} \omega^j \omega^{*n+1}_k = a^{-1} \coprod^*. \qquad (15)$$

From (15) follows that the rank of the second fundamental form is invariant under the choice of a frame. One says that a holomorphic hypersurface $M^n \in \mathbf{A}^{n+1}$ is non-degenerate if the second fundamental form is non-degenerate. Let us assume that $M^n$ is non-degenerate from now on.

Set

$$h_{ij} = \coprod(e_i, e_j) \quad \text{and} \quad H = \det(h_{ij}) \neq 0.$$

From (15) one gets

$$\begin{aligned} h_{ij} &= h_{ij} \omega^i \omega^j (e_i, e_j) \\ &= a^{-1} h^*_{kl} \omega^{*k} \omega^{*l} (e_i, e_j) \\ &= a^{-1} h^*_{kl} \omega^{*k} \omega^{*l} (b^p_i e^*_p, b^q_j e^*_q) \\ &= a^{-1} b^k_i b^l_j h^*_{kl}. \end{aligned} \qquad (16)$$

Hence,

$$H^* = a^{n+2} H. \qquad (17)$$

(17) implies that

$$(H^*)^{\frac{1}{n+2}} = a H^{\frac{1}{n+2}} \theta,$$

where $\theta$ is an $(n+2)$-nd root of unity. We define a normalized second fundamental form $\hat{\mathrm{II}}$ relative to the frame $(x, e_\alpha)$

$$\hat{\mathrm{II}} = \theta^{-1} H^{\frac{-1}{n+2}} \mathrm{II}.$$

Let $\hat{\mathrm{II}}^*$ be the normalized second fundamental form for the frame $(x, e_\alpha{}^*)$. Then from (15) and (17),

$$\begin{aligned} \hat{\mathrm{II}}^* &= \theta^{*-1} H^{*\frac{-1}{n+2}} \mathrm{II}^* \\ &= \theta^{*-1} a^{-1} H^{\frac{-1}{n+2}} \theta^{-1} a\, \mathrm{II} \\ &= \theta^{*-1} \theta^{-1} H^{\frac{-1}{n+2}} \mathrm{II} \\ &= \bar{\theta} \hat{\mathrm{II}}, \end{aligned} \tag{18}$$

where $\bar{\theta}$ is an $(n+2)$-nd root of unity.

Taking the exterior derivative of (8), one gets

$$d\omega_i^{n+1} = d(h_{ik}\omega^k) = dh_{ik} \wedge \omega^k + h_{ik}\omega^j \wedge \omega_j^k. \tag{19}$$

On the other hand, from (5),

$$d\omega_i^{n+1} = \omega_i^\alpha \wedge \omega_\alpha^{n+1} = \omega_i^l \wedge (h_{lm}\omega^m) + \omega_i^{n+1} \wedge \omega_{n+1}^{n+1} \tag{20}$$

Comparing (19) and (20), one gets

$$(dh_{ik} - h_{ij}\omega_k^j - h_{\ell k}\omega_i^\ell + h_{ik}\omega_{n+1}^{n+1}) \wedge \omega^k = 0. \tag{21}$$

Applying Cartan's lemma to (21), one gets