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NUCLEAR REACTOR THEORY

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EDITORS' PREFACE

The current era has been described as "the atomic age", and it seems probable that mankind will depend increasingly on nuclear energy during the next century. In the design of nuclear reactors, mathematical analysis already plays an important role.

Nevertheless, very few research mathematicians have so far devoted serious effort to the mathematical problems of nuclear reactor theory. The present volume is intended to increase the number of such mathematicians, by indicating the great variety of interesting mathematical problems encountered in this fascinating field. As a by-product, it may help to put the design of future nuclear reactors on a more scientific basis.

The contributors to this volume having already done their part, we hope that both pure and applied mathematicians will accept the challenging invitation offered, thereby continuing the great tradition of Archimedes, Newton, Gauss, Fourier, Maxwell, Poincaré, and many others. In this great tradition, each new major field of physical application has both suggested fundamental new mathematical concepts, and has owed much of its deeper development to the rigorous mathematical analysis of these concepts.

GARRETT BIRKHOFF
EUGENE P. WIGNER

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REACTOR TYPES

BY

ALVIN M. WEINBERG[1]

Ordinary fire has innumerable embodiments—from small-burning matches to vast central power plant boilers, from slow-simmering cigarettes to oxygen-hydrogen explosions. Yet in every case there are certain invariant conditions that characterize the phenomenon as fire—namely, reaction with oxygen, and a reaction rate great enough to maintain the temperature at a high level, essentially independent of the temperature of the surroundings.

So it is with neutron chain reactors or "nuclear" fires. As fuel for the fire one can choose U^{233}, U^{235}, Pu^{239}; as moderator, any light element which does not absorb neutrons; as coolant, almost any fluid. By choosing different combinations of constituents, or different arrangements of the constituents, one can arrive at perhaps 1000 different chain-reacting configurations, of which at least 100 probably are feasible from an engineering standpoint. In spite of this diversity, all neutron chain reactions have an essential common property. In every case neutrons induce the basic energy-liberating fission reaction, and they are themselves produced by the fission reaction. It is this property that gives to nuclear reactors their name—"chain reactors"—and to the mathematical theory of nuclear chain reactors a beautiful unity, only superficially obscured by less essential elements of diversity. In the following remarks, which are intended to give a background for this Symposium on Nuclear Reactor Theory, I shall first indicate how the most general, i.e., the chain, character of these devices leads to a very general mathematical theory of chain reactors, and I shall then indicate how the general theory becomes specialized in different directions for reactors of different engineering types.

The general chain reactor equation—kernel form. We consider a region of space occupied by a medium in which a chain reaction is taking place. The medium is characterized by a capture cross section $\Sigma_a(x,E)$ which depends on position x and energy E; a fission cross section $\Sigma_f(x,E)$; a neutron multiplicity $\nu(E)$ which gives the number of neutrons per fission induced by a neutron of energy E; a normalized[2] fission spectrum $f(E)$ which gives the fraction of fission neutrons emitted in unit energy around E[3]; and a general

[1] Oak Ridge National Laboratory, operated by Union Carbide Corporation for the U.S. Atomic Energy Commission.

[2] That is, $\int f(E)dE = 1$.

[3] In general $\nu(E)$ and $f(E)$ will vary from point to point in the reactor if the ratio of concentrations of fissionable species is not the same throughout the reactor—as in

transport kernel $H(x,x',E,E'\Omega,\Omega', t - t')$ which gives the flux of neutrons at time t in unit volume at x, and unit energy interval at E, whose velocity vectors lie in unit element of solid angle at Ω due to a single neutron produced (by fission or some other process) at (x',E',Ω',t'). For the moment we consider all of these quantities to be given, either experimentally, or as the result of auxiliary calculations.

Reactor theory, in highest approximation, concerns itself with the angular neutron flux, $\Phi(x,E,\Omega,t)$—i.e., the sum of the speeds of all the neutrons which, at time t, are in unit volume at x and unit energy interval at E and whose velocity vectors lie on unit solid angle at Ω. The integral of Φ over all directions

$$\Phi_0(x,E,t) = \int \Phi(x,E,\Omega,t)d\Omega$$

is called the scalar or total flux; if n is the number of neutrons per unit volume and unit energy E, then $\Phi_0 = nv$, where v is the speed corresponding to E. If the flux is independent of Ω, $\Phi(x,E,\Omega,t) = (1/4\pi)nv$. The first angular moment of Φ—i.e., the coefficient of the P_1 term in the expansion of Φ into spherical harmonics—is a three-component vector called the current, J; the components of J are

$$J_x = \int \Phi(x,E,\Omega,t)\Omega_x d\Omega, \quad J_y = \int \Phi(x,E,\Omega,t)\Omega_y d\Omega, \quad J_z = \int \Phi(x,E,\Omega,t)\Omega_z d\Omega.$$

We suppose that there are no extraneous sources, and that fission neutrons are produced isotropically. Then the number of fission neutrons produced in unit interval around (x,E,Ω,t) is

$$f(E) \int \frac{1}{4\pi} \nu(E'')\Sigma_f(x,E'')\Phi_0(x,E'',t)dE'';$$

hence, from the definition of the kernel H, the angular flux must satisfy

$$(1) \quad \Phi(x,E,\Omega,t) = \frac{1}{4\pi} \int_{-\infty}^{t} \int_{\Omega'} \int_{x'} \int_{E''} \int_{E'} \nu(E'')\Sigma_f(x',E'')\Phi_0(x', E'',t')f(E') \times H(x,x',E,E',\Omega,\Omega', t - t')dt'd\Omega'dx'dE''dE'.$$

Equation (1) is the general kernel form of the reactor equation for the angular flux under the restriction of no extraneous sources, constant ratio of fissionable species, and isotropic emission of fission neutrons.

In general, Equation (1) will not admit of any time-independent solutions except the trivial one $\Phi \equiv 0$. To cast (1) into the form of the more usual characteristic value problem, we introduce a fictitious neutron multiplicity,

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reactors which burn U^{235} and produce Pu^{239}. However, this is an unessential complication and for simplicity we assume the ratio of fissionable species to be constant. The fission spectrum $f(E)$ may depend upon the energy of the neutron causing fission, but this dependence will also be ignored since it is very weak.

$\nu_0(E)$, which is so chosen that Equation (1) has a stationary, nontrivial solution if the actual neutron multiplicity is replaced by the fictitious $\nu_0(E)$. For simplicity, we shall assume that the ratio $\nu(E)/\nu_0(E)$ is a constant C; this constant is called the static criticality constant or criticality factor. If the actual neutron multiplicity is greater than the multiplicity required for achievement of a stationary state, ($C > 1$), the neutron flux will grow—the reactor is supercritical; if $C < 1$, the flux will decrease and the reactor is subcritical; and if $C = 1$ the reactor is critical.

The critical reactor is stationary, i.e., $\Phi(x,E,\Omega,t)$ in the critical state does not depend on t. Hence, for this reactor we have

$$(2)\quad \Phi(x,E,\Omega) = \frac{1}{4\pi}\int\int\int \nu_0(E'')\Sigma_f(x',E'')\Phi_0(x',E'')f(E')K(x,x',E,E',\Omega)dx'dE'dE''$$

where

$$(3)\qquad K(x,x',E,E',\Omega) = \int_{\Omega'}\int_{-\infty}^{t} H(x,x',E,E',\Omega,\Omega'', t - t')dt'd\Omega'.$$

Thus the problem of determining the critical flux distribution and the value of the static criticality constant reduces to the solution of the characteristic value problem

$$(4)\quad \Phi(x,E,\Omega) = \frac{1}{4\pi C}\int\int\int \nu(E'')\Sigma_f(x',E'')\Phi_0(x',E'')f(E')K(x,x',E,E',\Omega) \times dx'dE'dE'',\ \Phi_0(x,E) = \int \Phi(x',E,\Omega)d\Omega$$

or, if we define the integral over Ω of $K(x,x',E,E',\Omega)$ by

$$(3a)\qquad K_0(x,x',E,E') = \int K(x,x',E,E',\Omega)d\Omega$$

then we can find the integral equation satisfied by the flux simply by integrating (4) over Ω:

$$(4a)\quad \Phi_0(x,E) = \frac{1}{4\pi C}\int\int\int \nu(E'')\Sigma_f(x',E'')\Phi_0(x',E'')f(E')K_0(x,x',E,E') \times dx'dE'dE''.$$

The flux at (x,E) coming directly—i.e., without intervening fissions—from a single fission neutron produced at x' is

$$(3b)\qquad \Phi_0(x,x',E) = \frac{1}{4\pi}\int f(E')K_0(x,x',E,E')dE'.$$

In the steady state this flux disappears either by absorption, or by escape

from the edges of the reactor. If we call the ratio of neutrons absorbed per second within the reactor to neutrons produced at x' per second the non-escape probability, $P(x')$, then, from (3b)

$$P(x') = \int\int \Sigma_a(x,E)\Phi_0(x,x',E)dxdE \tag{3c}$$

$$= \frac{1}{4\pi}\int\int\int \Sigma_a(x,E)f(E')K_0(x,x',E,E')dxdE'dE.$$

Neutrons cannot escape from an infinitely large reactor: hence, in an infinitely large reactor, $P(x') \equiv 1$, i.e.,

$$\frac{1}{4\pi}\int\int\int \Sigma_a(x,E)f(E')K_0(x,x',E,E')dxdE'dE = 1 \tag{3d}$$

(infinite critical system)

which simply says that, in the stationary state in an infinitely large critical reactor, the number of neutrons produced per second equals the number absorbed per second. Of course, since (3d) refers to an infinitely large system, the kernel K_0 in (3d) is not the same as the kernel K_0 in (3c) which refers to a system of finite size.

The explicit energy and angular dependence in the characteristic value problem (4a) can be eliminated, as first suggested by G. Placzek and G. Volkoff [1] by introducing as variable, $S(x)$, the number of fission neutrons produced per unit time at x:

$$S(x) \equiv \int \nu(E'')\Sigma_f(x,E'')\Phi_0(x,E'')dE''.$$

Then, upon multiplying (4a) through by $\nu(E)\Sigma_f(x,E)$ and integrating over E, (4a) becomes

$$S(x) = \frac{1}{C}\int S(x')G(x,x')dx' \tag{5}$$

where

$$G(x,x') = \frac{1}{4\pi}\int\int f(E')\nu(E)\Sigma_f(x,E)K_0(x,x',E,E')dEdE' \tag{6}$$

is the number of fission neutrons produced at x due to a single fission neutron produced at x'. For the moment we simply observe that for a uniform, infinitely large reactor,

$$\int G(x,x')dx = \frac{1}{4\pi}\int\int\int f(E')\nu(E)\Sigma_f(E)K_0(x,x',E,E')dxdE'dE \tag{6a}$$

(infinite medium)

or, according to (3d)

$$\int G(x,x')dx = \frac{\frac{1}{4\pi}\int\int\int f(E')\nu(E)\Sigma_f(E)K_0(x,x',E,E')dx\,dE'\,dE}{\frac{1}{4\pi}\int\int\int \Sigma_a(E)f(E')K_0(x,x',E,E')dx\,dE'\,dE} \tag{6b}$$

(infinite medium)

which is the number of neutrons produced per neutron absorbed. This quantity is called the infinite medium multiplication constant and is denoted by k. Once the source distribution $S(x)$ has been found by solution of the energy and angular independent (5), the full energy and angular dependent stationary flux $\Phi(x,E,\Omega)$ can be obtained by quadrature from (4), namely

$$4\pi\Phi(x,E,\Omega) = \frac{1}{C}\int\int S(x')f(E')K(x,x',E,E',\Omega)dE'dx', \tag{7}$$

or

$$\Phi_0(x,E) = \frac{1}{4\pi C}\int\int S(x')f(E')K_0(x,x',E,E')dE'dx'. \tag{7a}$$

The foregoing discussion is almost entirely formal. Perhaps the simplest physically significant restriction on the kernels H, K and G is that they are always positive for all values of their arguments—this merely means that the neutron fluxes are positive quantities. It is remarkable that this almost trivial restriction on the kernels—a restriction common to all reactors—already is sufficient to make a fruitful spectral theory possible; and in the papers by Birkhoff, Varga, Habetler and Martino, and Wing very general properties of integral equations with positive kernels will be discussed. Perhaps the most important result is that in every non-critical reactor the neutron distribution finally is dominated by the so-called "persisting distribution"—i.e., a distribution which varies exponentially

$$\Phi_0(x,E,t) = \Phi_{p_0}(x,E)e^{\lambda t}, \tag{8}$$

and which remains after all other higher spatial modes have decayed.[4] If the reactor is supercritical, $\lambda > 0$ (and $C > 1$); if it is subcritical, $\lambda < 0$ (and $C < 1$); and if it is just critical, $\lambda = 0$ (and $C = 1$). It is therefore reasonable to expect $C - 1$ to be proportional to the time constant λ; under quite general conditions which we shall not investigate, it is indeed true that

$$\frac{C-1}{\bar{l}} = \lambda \tag{9}$$

where $\bar{l}$ is the so-called "mean generation time" of the neutron in the reactor.

[4] If there are delayed neutrons then the persisting distribution will vary in time as a sum of exponentials rather than as a single exponential as in (8).

Alternately, $\bar{l}$ can be considered to be defined by (9); in this case $\bar{l}$ depends slightly on ν.

The Boltzmann form of the reactor equations. The foregoing formulation of reactor theory puts the entire substance of the theory into the all-powerful and rather mysterious kernel H, or the derived kernels—only slightly less powerful and mysterious, K, K_0, and G. Actually the kernel G, at least, can be determined experimentally if the medium is uniform, isotropic, and infinitely large. In that case, all the kernels, besides being essentially positive, are displacement operators in $|x - x'|$. However, if the reactor is finite, or non-isotropic, or non-uniform, then the experimental measurement of the kernels is much too unwieldy to be practical. It is therefore necessary to find some general mathematical equation which will allow the computation of the kernels, or at least allow further deductions about their properties. Most of reactor theory is based on the assumption that the kernel $H(x,x', E,E',\Omega,\Omega', t - t')$ satisfies a time-dependent equation involving a linear operator Q:

$$\frac{\partial H}{\partial t} + QH = \delta(x - x')\delta(E - E')\delta(\Omega - \Omega')\delta(t - t') \tag{10}$$

with initial condition $H = 0$ for $t \leqq t'$. The linear operator Q, which in reactor theory is almost always non-Hermitian, may involve all of the variables (x,E,Ω), but *not* the time. Hence the general reactor theory discussed in most of the following papers is confined to situations in which the *properties* of the reactor, though not the fluxes, remain independent of time.

Equation (10) is a symbolic way of writing the Boltzmann equation[5] for the kernel H; however, in the case where delayed neutrons are considered, it can be used to determine those components of H which are associated with the delayed neutron emitters. In this respect (10) is more general than the Boltzmann equation which ordinarily deals only with the neutron fluxes, not the density of delayed neutron emitters.

By means of (10) it is a simple matter to obtain the Boltzmann form of (1). Differentiation of (1) with respect to t gives

$$\begin{aligned}\frac{\partial \Phi(x,E,\Omega,t)}{\partial t} = \lim_{t' \to t} \iiiint \nu(E'')\Sigma_f(x',E'')\Phi_0(x',E'',t')f(E') \\ \times H(x,x',E,E',\Omega,\Omega', t - t')d\Omega' dx' dE'' dE' \\ + \frac{1}{4\pi} \iiiint \nu(E'')\Sigma_f(x',E'')\Phi_0(x',E'',t')f(E') \\ \times \frac{\partial H(x,x',E,E',\Omega,\Omega', t - t')}{\partial t} dt' d\Omega' dx' dE'' dE'.\end{aligned} \tag{1a}$$

[5] The Boltzmann equation is given explicity, e.g., in the papers by Professor Wigner and by Dr. Wilkins in this symposium.

The first term on the right-hand side of (1a) vanishes because $H = 0$ for $t \leqq t'$. The second term may be transformed by replacing $\partial H/\partial t$ by its equivalent value, $\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(E - E')\delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}')\delta(t - t') - QH$, given in (10). Since Q depends only on $\boldsymbol{x}$, E, and $\boldsymbol{\Omega}$, it may be taken out of the integral, and one obtains, on using (1), the Boltzmann form of the reactor equation

$$(11)\quad \frac{\partial \Phi(\boldsymbol{x},E,\boldsymbol{\Omega},t)}{\partial t} + Q(\boldsymbol{x},E,\boldsymbol{\Omega})\Phi(\boldsymbol{x},E,\boldsymbol{\Omega},t) = \frac{f(E)}{4\pi}\int \nu(E')\Sigma_f(\boldsymbol{x},E')\Phi_0(\boldsymbol{x},E',t)dE',$$

$$(11a)\quad \Phi_0 \equiv \int \Phi(\boldsymbol{\Omega})d\boldsymbol{\Omega}.$$

Equation (11) has the form of a characteristic value problem whereas (10) does not—i.e., the kernel H characterizes the medium, so to speak, in all of its *non-multiplying* aspects; it is the multiplying or *chain* character of the reaction in the medium which introduces the homogeneous linear term on the right-hand side of (11), and thus makes the basic reactor equation for all reactors a characteristic value problem. Much of the discussion to be given over the next two days will be based on Equation (11) or a variant thereof.

The advantage of the Boltzmann formulation (11) is that it deals directly with the fluxes, and so can be applied, in principle, directly to any multiplying medium, finite, infinite, uniform, or non-uniform. For computational purposes, it is just about as difficult, in the general case, to deal directly with Φ [as in (11)] as it is to bother about H as in (1) or (10). Of course (1) and (11) are equivalent —(1) is the resolved or integral form of (11).

Energy dependence of the fluxes. The foregoing remarks have been completely general and are applicable to every reactor. Fortunately, the completely general theory is rarely, if ever, used—instead certain simplifications and schematizations are always possible which make the theory at once more tractable and more transparent. The nature of the approximations are different for different types of reactors. The customary approximate models for dealing with each of the variables—energy, space, angle, and time—in different types of reactors will be examined one by one even though the justification for considering the energy, spatial, angular, and temporal dependence separately is rigorous only if the reactor is uniform and infinitely large. In that case the angular flux $\Phi(\boldsymbol{x},E,\boldsymbol{\Omega},t)$ finally approaches a separable distribution—i.e., a product of functions of each of the variables $\boldsymbol{x}$, E, $\boldsymbol{\Omega}$, and t. Even in finite, but otherwise uniform reactors, the stationary distribution of $\Phi(\boldsymbol{x},E,\boldsymbol{\Omega})$ is a product of a function of $(\boldsymbol{x},\boldsymbol{\Omega})$ and a function of E in those regions of the reactor which are far from boundaries—this is the substance of the so-called first fundamental theorem of reactor theory. On the other hand, if the reactor is non-uniform, or is very small, the distributions are not separable even in $(\boldsymbol{x},\boldsymbol{\Omega})$ and E. Nevertheless, it is profitable and suggestive to consider the energy distribution separately from the spatial and angular

distributions even in those cases where the stationary distribution is not really separable.

From the point of view of the energy dependence of Φ, reactors can be classified roughly into fast, intermediate or resonance, and thermal reactors. These reactor classes differ primarily in the ratio of heavy fissionable fuel nuclei to light moderator nuclei which they contain. In general, this ratio is high in fast reactors (e.g., 1 atom of U^{235} to no atoms of H), low in thermal reactors (e.g., 1 atom of U^{235} to 1000 atoms of H), and in between in intermediate or resonance reactors. The difference in composition implies an essential difference in the mechanism of energy degradation or moderation of the neutrons: in fast reactors, the mechanism of moderation is primarily inelastic scattering with the heavy fuel nuclei; in thermal reactors, it is primarily elastic scattering with the light moderator nuclei.

Since inelastic scattering becomes ineffective as an energy loss mechanism below, say, 300 kev, fast reactors are reactors in which the neutron flux, Φ_0, and the fission rate, $\Sigma_f\Phi_0$, are confined to an energy band rather close to the energy range covered by the spectrum of fission neutrons, $f(E)$. The fission spectrum has a maximum around 2 Mev, and falls to nearly zero at 1/2 Mev on the low side, perhaps 10 Mev on the high side—that is, the fission spectrum occupies about 3 units on the lethargy scale.[6] The neutrons in a fast reactor, once they fall below the energy at which inelastic scattering is ineffective, wander about, *without much change in energy*, until they are captured or escape from the boundary of the reactor, or until they cause new fissions. It is for this reason that the neutrons in fast reactors occupy such a relatively small lethargy interval.

Since the lethargy interval occupied by neutrons is so small, it is often a surprisingly good approximation to assume in fast reactors that all the neutrons have the same energy, E_0. This corresponds to assuming that both the fission spectrum and the kernels H and K are δ-functions in E:

$$f(E') = \delta(E_0 - E'), \qquad K_0(x,x',E,E') = 4\pi\delta(E' - E)\cdot T(x,x')$$

where T is a function only of x and x'. Hence in zeroth order, Equation (4a), which the stationary distribution in a fast reactor obeys, reduces to

$$\Phi_0(x) = \int \nu\Sigma_f\Phi_0(x')T(x,x')dx' \tag{12}$$

where the parameters ν and Σ_f are evaluated at the energy E_0. A first-

[6] Since, in the moderation of neutrons by elastic collision, the neutron loses a certain *fraction* of its energy at each collision, it is customary to measure energy on the *lethargy* scale—the lethargy u of a neutron of energy E (in Mev) is defined as $u = \ln 10/E$; thus in fast reactors the entire neutron flux is confined to a lethargy interval of about $\ln 10/(1/2) = \ln 20 \cong 3$ units. This may be compared with the lethargy interval of about 18 to 20 units which is occupied by the neutron distribution in reactors in which there is a large amount of moderator mixed with the fissionable fuel.

order improvement on (12) is to give $f(E)$ its actual shape, but to assume that there is no neutron moderation—i.e., the kernel K_0 is still a δ-function of $(E - E')$. In that case the energy dependence of Φ_0 is also given by $f(E)$, as can be seen from (4a), and the kernel $G(x,x')$ in (6) is merely

$$G(x,x') = T(x,x') \int f(E)\nu(E)\Sigma_f(E)dE;$$

i.e., in effect the $\nu\Sigma_f$ in (12) is averaged over the fission spectrum. A further improvement would be to take explicit account of the energy degradation due to inelastic scattering, but this refinement will not be pursued further.

At the other extreme are thermal reactors—for example, a dilute solution of U^{235} in ordinary or heavy water. In such reactors the fissionable isotope is diluted with so much moderating material (there may be as many as 2000 H atoms per U^{235} atom) that collision with the moderator is by far the most likely process until the neutron has escaped from the reactor or has become thermal. Once the neutron has become thermal, moderation ceases, and the neutron diffuses without energy change until its life is ended by capture or escape as a thermal neutron.

Thus in a thermal reactor the neutron spectrum spans the entire lethargy range of about 20 units; in addition, the thermal energy range is also populated. However, although there are neutrons at every energy, because the fission cross section is much higher at thermal energy than at higher energies, the spectrum of neutrons which induces fissions is very nearly a δ-function at thermal energies—i.e., $\Sigma_f(x,E) \sim \Sigma_{ft}(x)\delta(E - E_t)$, where Σ_{ft} is the cross section for fission by thermal neutrons. Hence, again the theory acquires an important simplicity—the function $G(x,x')$ becomes

$$\text{(12a)} \qquad G(x,x') = \nu_t\Sigma_{ft}(x) \int f(E')K_0(x,x',E_t,E')dE',$$

the subscript t denoting that the quantities are evaluated at thermal energies. Equation (12a) is a good deal simpler than (6) since it involves an integration only over the variable E' instead of over both E and E'. Now the kernel $\int K_0(x,x',E_t,E')f(E')dE'$ could be determined by allowing a foil at point x which is sensitive to thermal neutrons to be activated by a point source of fission neutrons at x' in a medium which has the same moderating and absorbing properties as well as geometric shape as the reactor. This measurement is still complicated because it must be done for every x,x'. Nevertheless there is the essential simplification that K_0 need be known only at energy E_t: only thermal neutrons enter the multiplication process, and therefore only the distribution of thermal neutrons is significant.

Actually the situation, even in thermal reactors, is more complicated. In the first place, even though moderation by elastic collision is the predominant nuclear process affecting fast neutrons, some fast neutrons do react with fuel and cause fission. This effect is particularly noticeable in reactors whose

fuel is unenriched or only slightly enriched uranium. The U^{238} can undergo fission if the incident neutrons have energy in excess of the fission threshold—about 1.1 Mev. Thus the spectrum of neutrons which produces fissions in natural-uranium, thermal reactors has two peaks—a very large peak (accounting for better than 97% of the fissions) caused by thermal neutron-induced fission in U^{235}, and a much smaller peak (accounting for less than 3% of the fissions) caused by fast neutron-induced fission in U^{238}. There is of course some fission between these two peaks—however, it is very small since the U^{235} fission cross section is relatively small in the intermediate region. The fast neutron fission in U^{238} causes the so-called "fast effect"; it may be accounted for by imposing on $G(x,x')$ the form

$$(13)\quad G(x,x') = \nu_t\Sigma_{ft}\int_{E'} f(E')K_0(x,x',E_t,E')dE' + \overline{\nu_f\Sigma_{ff}}\int_{\substack{E>1.1\\ \text{Mev}}}\int_{E'} f(E')K_0(x,x',E,E')dE'dE$$

where subscript f on a quantity means that the quantity refers to fast neutrons and the bar over $\overline{\nu_f\Sigma_{ff}}$ means that this quantity is averaged over the spectrum which is effective in producing fast fissions in U^{238}. The simplification in $G(x,x')$ of (13) as compared to the general expression for G given in (6) is that the integration over both E' and E in the second term extends over so small a lethargy interval that the kernel K_0 can fairly well be assumed to be independent of energy.

On the other hand, the value of the second integral is very sensitive to the manner in which the fuel is disposed. If the fuel is in the form of discrete lumps fission neutrons will be much more likely to encounter U^{238} and cause fission than if the fuel is uniformly divided. Thus the fast effect—i.e., essentially the second term in (13)—is considerably greater in lumped (heterogeneous) systems than in dispersed (homogeneous) systems.

A second essential complication is that some capture does occur during moderation even in thermal reactors. In the original graphite-moderated, natural-uranium reactors about 10% of the neutrons are captured in U^{238} and do not cause fission during moderation. This "resonance capture" occurs mainly at discrete resonance energies; its general effect is to diminish the flux of neutrons as the neutrons lose energy by elastic collision. Thus one factor of the kernel $K_0(x,x',E,E')$ is the "resonance escape probability", $p(E,E')$, which gives the probability that a neutron whose initial energy is E' will escape resonance capture in slowing down to energy E. The logarithm of the function $p(E,E')$, under quite general assumptions, has the form

$$(14)\quad \ln p(E,E') \sim -\int_E^{E'} \Sigma_{a_{eff}}(E)\frac{dE}{E}$$

where $\Sigma_{a_{eff}}$ is an abbreviation for the so-called "effective" absorption cross

section of U^{238} and other resonance absorbers. The word "effective" is used here because the absorption depends not only upon the energy variation of the actual absorption cross sections; it depends also on the ratio of scattering to absorption cross section, and upon the gross geometric form of the absorber. The calculation of $\Sigma_{a_{eff}}(E)$, as well as its measurement, has received a great deal of attention since the beginning of chain reactor technology, and the present status of the calculation of $\Sigma_{a_{eff}}(E)$ will be summarized in a paper by L. W. Nordheim.

A third complication in thermal reactors is that the schematic assumption that the energy spectrum of thermal neutrons is a δ-function or even that the neutrons are in Maxwellian distribution is in many cases inadequate. Neutrons become thermal by slowing down from higher energy; they are removed from the thermal region either by capture or by escape from the reactor. The energy distribution of the neutrons is determined by the details of the mechanism of energy exchange between the moderator and the neutrons and by the balance between the rate at which the neutrons enter the thermal energy region, and the rate at which they are removed from the thermal region. Again, a great deal of work has been done on the calculation and the measurement of the spectrum of thermal neutrons in a reactor; this will be the subject of the paper by M. S. Nelkin.

It is much more difficult to make general remarks about intermediate reactors—first, because the spectrum of neutrons which induce fissions can extend over the whole lethargy scale of 20 units, and can include the thermal energy region; and second, because in these reactors, which are usually neither large nor uniform, the flux is anything but separable in space and energy. The only really simple statement about the energy dependence in such reactors is that, if the absorption or leakage is not very strong, the energy dependence of collision density (i.e., the product of total cross section and flux, $\Sigma\Phi_0$) is proportional to $1/E$:

$$\Sigma\Phi_0 \sim \frac{1}{E}. \tag{15}$$

A more accurate representation of the collision density is

$$\Sigma\Phi_0 \sim \frac{1}{E}\int p(E,E')f(E')P(E,E')dE' \tag{15a}$$

where $p(E,E')$ is the resonance escape probability given in (13), and $P(E,E')$ is the probability that a neutron originating at energy E' will reach energy E before escaping from the reactor.

These statements are, however, only approximate. What is perhaps most relevant, mathematically, for intermediate reactors, is to have a convenient and tractable approximation scheme for computing the energy distribution even when the flux is not separable into energy, space, and angle—in other words, a practical way to find solutions of (11). The commonest such scheme is the so-called "multigroup" method. This method merely amounts