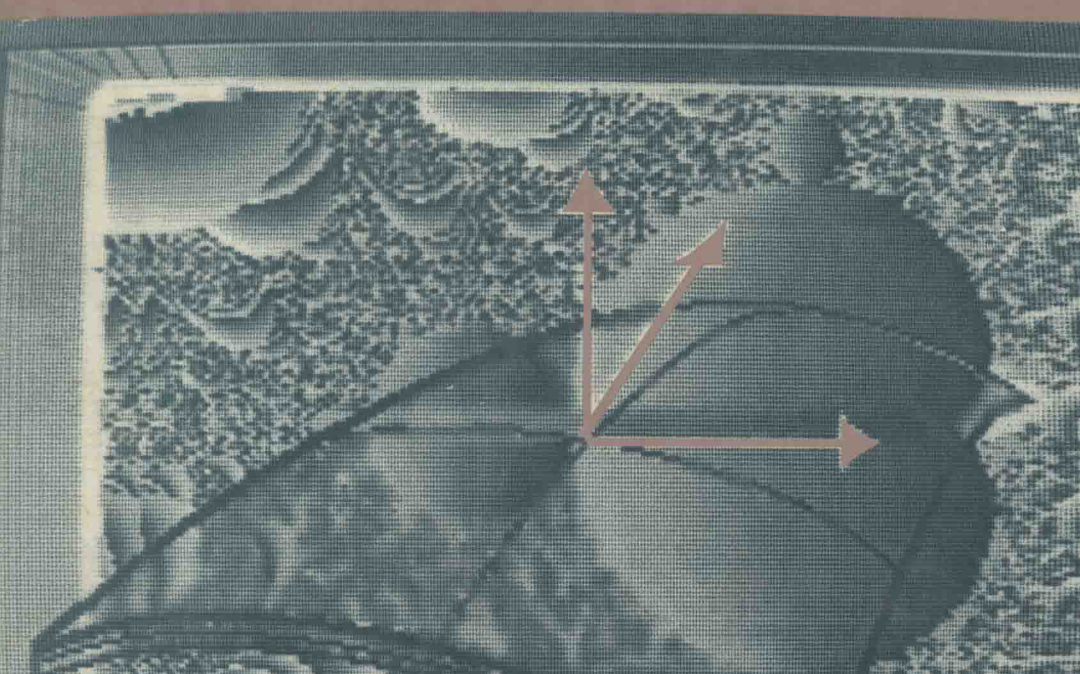
AN INTRODUCTION TO THE
MATHEMATICS AND GEOMETRY

Computer Graphics



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Preface

Computer Graphics: An Introduction to the Mathematics and Geometry introduces the mathematical and geometric principles supporting computer graphics and a large class of applications referred to as geometric modeling. This textbook is intended for the lower-division college student majoring in computer science, engineering, or applied mathematics whose special interests are in computer graphics, CAD/CAM systems, geometric modeling, or related subjects. *Computer Graphics* can also serve as a supplement to upper-division and graduate-level courses or as a useful addition to the reference libraries of professionals.

Prerequisites include college-preparatory mathematics through trigonometry; an introduction to solid geometry would be useful. At least one course in computer programming would be helpful, since programming *per se* is not a subject of this textbook. Calculus is not required, although a prior or concurrent course in elementary calculus would be of benefit to understanding certain topics here. Some elementary calculus is introduced where tangents and normals to curves and surfaces are discussed.

A set of integrated concepts is presented whose mastery is necessary for a thorough understanding of the mathematics and geometry underlying computer graphics. Several important concepts are emphasized throughout this text. These concepts include parametric geometry; transformations; vectors; matrix methods; and, to a lesser extent (although nonetheless important), data structures, algorithm development, and computational efficiency.

The text is interspersed with "boxes"—one page or two facing pages—presenting tightly focused information on a specific problem elaborating on a concept in the text proper. Of course, there are many exercises, and the answers to some are provided. A suggestion: Read all the exercises (even those not assigned); they are a good sampling of problems that may be faced in a professional capacity.

Much of the work in Section 1.4, Display Coordinate Systems, and Chapter

16, Display and Scene Transformations, was inspired by the work of Bill Miller, whose IMAGINATOR three-dimensional graphics program effectively demonstrates the application of these concepts.

Thanks to James Geronimo and Woodrow Chapman of Industrial Press for their thoughtful and professional guidance throughout the preparation of this textbook.

Finally, thanks to my wife, Janet. This is the second time in five years that she has devoted much time and talent to the production of a very difficult manuscript. *Mahalo nui loa.*

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AN INTRODUCTION TO THE MATHEMATICS AND GEOMETRY

1. Coordinate Systems

GEOMETRIC OBJECTS have many important characteristics, such as size and shape, location and orientation, and certain spatial relationships to each other. To describe, measure, and analyze these characteristics consistently and quantitatively requires a frame of reference called a **coordinate system.** There are many kinds of coordinate systems, each offering special advantages depending on the problem to be solved. In fact, a coordinate system is, in and of itself, a kind of geometric object.

When you complete your studies in this chapter, you will be acquainted with a variety of coordinate systems and will begin to understand something of their application and versatility. You will learn that one kind of system can be transformed (changed) into another kind, and you will be prepared to work with the combinations of coordinate systems needed to generate computer graphic displays.

So now let us look at the coordinate systems used in computer graphics.

1.1. One-Dimensional Coordinate Systems

Perhaps the most important feature of a coordinate system is its number of dimensions. The simplest system is one dimensional and consists of an unbounded straight line (Fig. 1.1). A reference point, O, the **origin,** allows you to locate, construct, and measure geometric objects; in this system, objects are limited to points and lines. Locate a point, **p**, by giving its distance, x, from the origin. This single number, or **coordinate,** is sufficient to describe **p**. A plus or minus sign indicates the direction of the point relative to the origin. Mathematicians call this system the **real line,** because there is a one-to-one correspondence between the points on the line and the set of real numbers. (Note that a boldface, lowercase **p** denotes a geometric point. You will find this notation used throughout the

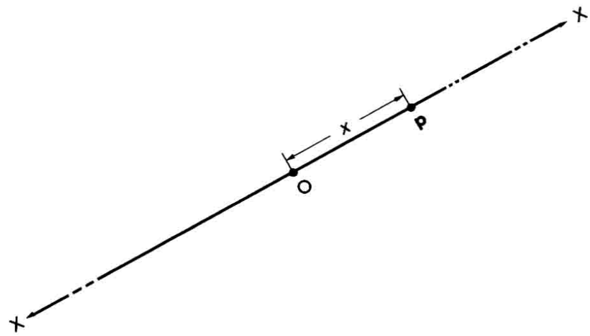

Fig. 1.1. One-dimensional coordinate system. (Note that coordinate systems need not be horizontal.)

text, with some minor exceptions usually related to algebraic considerations.)

The location of the origin is arbitrary. Choosing a new origin, O^*, does not change the geometric characteristics of objects defined in the system (Fig. 1.2). For example, the distance between points $\mathbf{p}_1$ and $\mathbf{p}_2$ is the same for either the O or O^* system. Let O and O^* be separated by a distance d. The coordinate of O^* in terms of the initial coordinate system, O, is $x_T = -d$. Then, in the O^* system, the coordinate of $\mathbf{p}_1$ is $x_1 + d$, and that of $\mathbf{p}_2$ is $x_2 + d$. This process of changing the origin is known as a **transformation.** Of course, O^* could just as easily be to the right of O. If that were the case, the two preceding expressions would not give correct results. To account for this, use the algebraic value of the coordinate of O^* in the O system, x_T. This produces the following correct expression relating the coordinate x of any point $\mathbf{p}$ in the O system to its coordinate x^* in the O^* system:

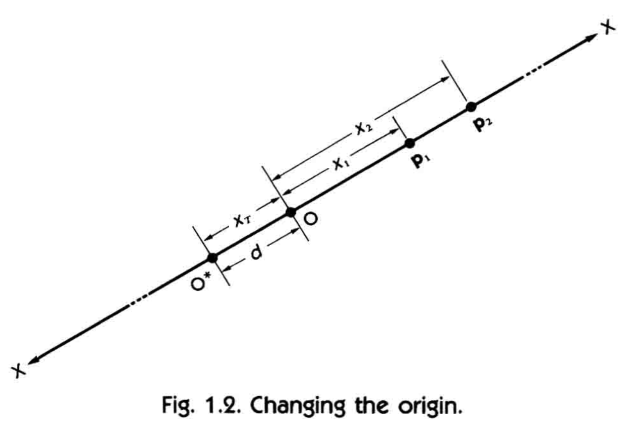

Fig. 1.2. Changing the origin.

$$x^* = x - x_T \tag{1.1}$$

The asterisk denotes the coordinate of **p** in the new or transformed system. Equation (1.1) is called a transformation equation. The concept of a transformation is an important one in geometric modeling and computer graphics. In your studies you will encounter many other coordinate system transformations as well as many other kinds of transformations.

One-dimensional coordinate systems are not the most exciting topics you will study, but they are not without some interesting properties, and they give rise to a rich assortment of concepts. For example, consider an interval on the real line, bounded by the points x_a and x_b (Fig. 1.3), where $x_a < x_b$. (It is easy to think of this interval as merely a line segment, but you should also try to keep in mind the concept of the interval as a set of continuous points.)

An arbitrary point, x, is either in this interval (a member of the set of points) or outside of it. If the bounding points x_a and x_b are declared to be in the interval (or set) and if $x_a \le x \le x_b$, then x is in the interval (or set). This is a **closed interval** because the boundary or limit points are included. An **open interval** looks like $x_a < x < x_b$; here the boundary points are not included. If you want to use more concise notation, then write

$$x \in [a, b] \quad \text{for a closed interval} \tag{1.2}$$

$$x \in (a, b) \quad \text{for an open interval} \tag{1.3}$$

Read the symbol $\in$ as "is an element (or member) of." This notation from set theory applies to many algebraic and geometric situations you will encounter. A small refinement of interpretation with considerable geometric significance is: If $x = a$ or $x = b$, then x is on the boundary.

Other examples of "geometry" on this line include relationships among points, points and lines, and lines. Does one point fall between two other points? Do two line segments overlap? What is the length of a line segment? You can discover many other properties and rules. Try some.

One-dimensional coordinate systems need not be limited to unbounded straight lines. Consider a semiinfinite line (or half line) or finite line segment. And then there are coordinate systems and geometries you could construct on closed curves. These more exotic systems are not just mental exercises; their usefulness is apparent in other branches of math-

Fig. 1.3. An interval on the real line.

ematics and in the physical sciences. Think of coordinate systems as geometric tools available to you to help you understand and solve many kinds of problems. But first, it is important that you understand them and their versatility, and how to use them to advantage.

EXERCISES

1. Given the interval defined by a set of points x such that $x \in [-3.5, 8]$, state whether the following points are inside or outside the interval:

a. (2)
b. (0)
c. (−4)
d. (7)
e. (9.5)
f. (3.5)
g. (−8)
h. (8.1)
i. (−1.5)
j. (4)

2. If line segments in a one-dimensional coordinate system are given by the interval notation, then describe the relationship of each of the following six lines to the other five:

a. $x_A \in [-3, -1]$
b. $x_B \in [4, 10]$
c. $x_C \in [12, 16]$
d. $x_D \in [-4, 1]$
e. $x_E \in [2, 5]$
f. $x_F \in [5, 7]$

3. Given the 10 points a–j of Exercise 1, state whether they are inside or outside each of the following four intervals:

a. $x_A \in [3, 9.5]$
b. $x_B \in (-4, 4)$
c. $x_C \in [0, 10.1]$
d. $x_D \in (-1, 3.5)$

4. Show that the distance between $\mathbf{p}_1$ and $\mathbf{p}_2$ in Fig. 1.2 is the same in both coordinate systems.

5. Describe some of the features of a closed, circular one-dimensional coordinate system. *Hint:* How would you locate points and intervals? What about segments longer than the circumference?

1.2. Two-Dimensional Coordinate Systems

Early in the 17th century, the French mathematician and philosopher René Descartes revolutionized the study of geometry. Descartes recog-

nized that algebraic functions could be interpreted geometrically by graphing them onto a two-dimensional coordinate system. This idea supported earlier notions of a one-to-one correspondence between algebra and geometry. Not only could algebraic equations be interpreted as graphed curves, but, also, many geometric figures were found to have simple algebraic equivalents. In fact, we now know, thanks to Descartes and others, that all geometric objects—curves, surfaces, and more exotic types—can be described algebraically. And all algebraic functions have a geometric interpretation. This large and important branch of mathematics is called **analytic geometry,** and Descartes is its founder. The rectangular, two-dimensional system of coordinates he used to graph the algebraic functions (although not invented by him) is called a **Cartesian coordinate system.**

Two unbounded straight lines intersecting at right angles form the **principal axes,** x and y, of the coordinate system. Their point of intersection, O, is the origin. A grid of equally spaced lines may be laid out parallel to the principal axes, although this is not strictly necessary. Now every point in the plane is located and defined by a pair of numbers (x, y), the coordinates of the point. In Fig. 1.4 the coordinates of the point $\mathbf{p}_1$ are (x_1, y_1). At x_1 on the x axis, construct a line parallel to the y axis. Similarly, at y_1 on the y axis, construct a line parallel to the x axis. These two lines intersect at $\mathbf{p}_1$.

Note that the principal axes do not necessarily have to be at right angles to each other; but if they are not, it is no longer a *Cartesian* system. Computations in a skewed-axis system are somewhat more complex, and trigonometric relationships are more cumbersome to express. The coor-

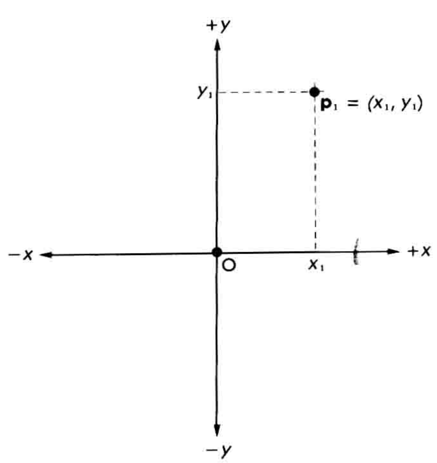

Fig. 1.4. Two-dimensional Cartesian coordinate system.