Lecture Notes in Mathematics

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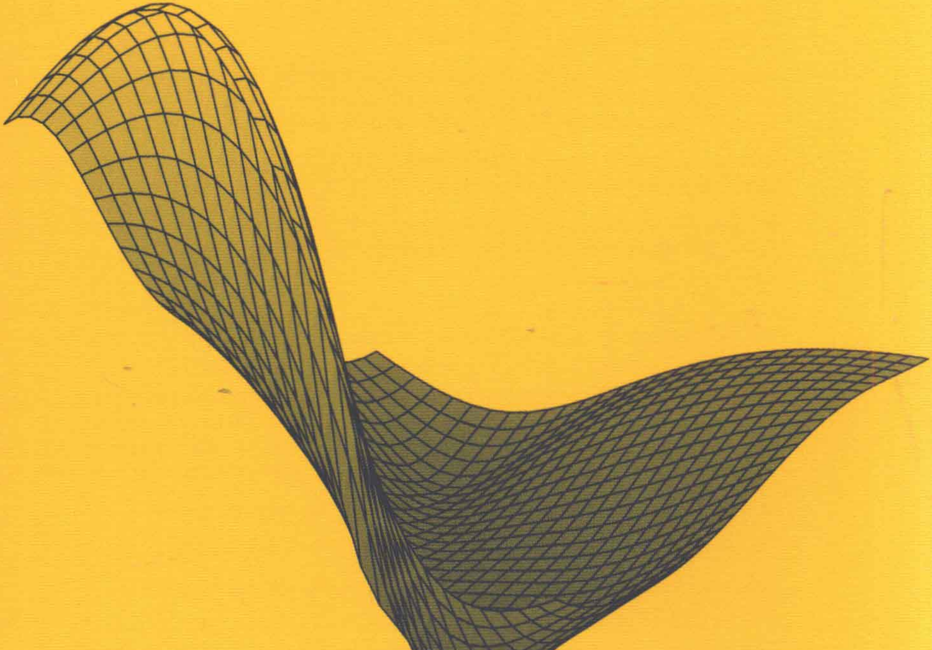
Peter Giesl

Construction of Global Lyapunov Functions Using Radial Basis Functions



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Author

Peter Giesl
Centre for Mathematical Sciences
University of Technology München
Boltzmannstr. 3
85747 Garching bei München
Germany
e-mail: giesl@ma.tum.de

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Lecture Notes in Mathematics

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Lecture Notes in Mathematics 1904

Preface

This book combines two mathematical branches: dynamical systems and radial basis functions. It is mainly written for mathematicians with experience in at least one of these two areas. For dynamical systems we provide a method to construct a Lyapunov function and to determine the basin of attraction of an equilibrium. For radial basis functions we give an important application for the approximation of solutions of linear partial differential equations. The book includes a summary of the basic facts of dynamical systems and radial basis functions which are needed in this book. It is, however, no introduction textbook of either area; the reader is encouraged to follow the references for a deeper study of the area.

The study of differential equations is motivated from numerous applications in physics, chemistry, economics, biology, etc. We focus on autonomous differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ which define a dynamical system. The simplest solutions $x(t)$ of such an equation are equilibria, i.e. solutions $x(t) = x_0$ which remain constant. An important and non-trivial task is the determination of their basin of attraction.

The determination of the basin of attraction is achieved through sublevel sets of a Lyapunov function, i.e. a function with negative orbital derivative. The orbital derivative $V'(x)$ of a function $V(x)$ is the derivative along solutions of the differential equation.

In this book we present a method to construct Lyapunov functions for an equilibrium. We start from a theorem which ensures the existence of a Lyapunov function T which satisfies the equation $T'(x) = -\bar{c}$, where $\bar{c} > 0$ is a given constant. This equation is a linear first-order partial differential equation. The main goal of this book is to approximate the solution T of this partial differential equation using radial basis functions. Then the approximation itself is a Lyapunov function, and thus can be used to determine the basin of attraction.

Since the function T is not defined at x_0, we also study a second class of Lyapunov functions V which are defined and smooth at x_0. They satisfy the equation $V'(x) = -p(x)$, where $p(x)$ is a given function with certain properties, in particular $p(x_0) = 0$.

For the approximation we use radial basis functions, a powerful meshless approximation method. Given a grid in $\mathbb{R}^n$, the method uses an ansatz for the approximation, such that at each grid point the linear partial differential equation is satisfied. For the other points we derive an error estimate in terms of the grid density.

My Habilitation thesis [21] and the lecture "Basins of Attraction of Dynamical Systems and Algorithms for their Determination" which I held in the winter term 2003/2004 at the University of Technology München were the foundations for this book. I would like to thank J. Scheurle for his support and for many valuable comments. For their support and interest in my work I further wish to thank P. Kloeden, R. Schaback, and H. Wendland. Special thanks to A. Iske who introduced me to radial basis functions and to F. Rupp for his support for the exercise classes to my lecture. Finally, I would like to thank my wife Nicole for her understanding and encouragement during the time I wrote this book.

December 2006 *Peter Giesl*

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1

Introduction

1.1 An Example: Chemostat

Let us illustrate our method by applying it to an example. Consider the following situation: a vessel is filled with a liquid containing a nutrient and bacteria, the respective concentrations at time t are given by $x(t)$ and $y(t)$. This family of models is called chemostat, cf. [56]. More generally, a chemostat can also serve as a model for population dynamics: here, $x(t)$ denotes the amount of the prey and $y(t)$ the amount of the predator, e.g. rabbits and foxes.

The vessel is filled with the nutrient at constant rate 1 and the mixture leaves the vessel at the same rate. Thus, the volume in the vessel remains constant. Finally, the bacteria y consumes the nutrient x (or the predator eats the prey), i.e. y increases while x decreases.

The situation is thus described by the following scheme for the temporal rates of change of the concentrations x and y:

- x (nutrient): rate of change = input − washout − consumption
- y (bacteria): rate of change = − washout + consumption

The rates of change lead to the following system of ordinary differential equations, where the dot denotes the temporal derivative: $\dot{} = \frac{d}{dt}$.

$$\begin{cases} \dot{x} = 1 - x - a(x)\,y \\ \dot{y} = -y + a(x)\,y. \end{cases} \tag{1.1}$$

The higher the concentration of bacteria y is, the more consumption takes place. The dependency of the consumption term on the nutrient x is modelled by the non-monotone uptake function $a(x) = \frac{x}{\frac{1}{16} + \frac{x}{4} + x^2}$, i.e. a high concentration of the nutrient has an inhibitory effect. The solution of such a system of differential equations is unique, if the initial concentrations of nutrient and bacteria, $x(0)$ and $y(0)$, respectively, are known at time $t = 0$.

Imagine the right-hand side of the differential equation (1.1) as a vector field $f(x, y) = \begin{pmatrix} 1 - x - a(x)\,y \\ -y + a(x)\,y \end{pmatrix}$. At each point (x, y) the arrows indicate the

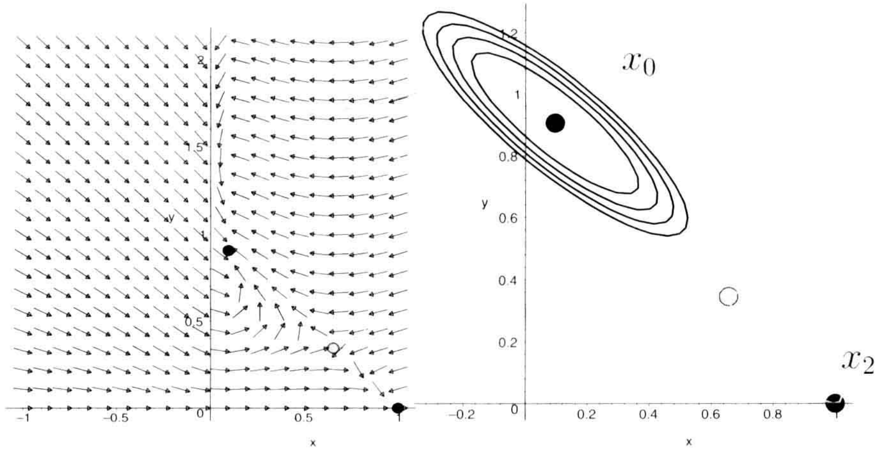

Fig. 1.1. Left: the vector field $f(x, y)$ (arrows with normalized length) and the three equilibria x_1 (unstable, grey), x_0 and x_2 (asymptotically stable, black). Right: the three equilibria and the local Lyapunov function $\mathfrak{v}$: the sign of the orbital derivative $\mathfrak{v}'(x, y)$ (grey) and the level sets $\mathfrak{v}(x, y) = 0.025, 0.02, 0.015, 0.1$ (black) which are ellipses. The sublevel sets are subsets of the basin of attraction $A(x_0)$.

infinitesimal rate of change, to which the solution is tangential, cf. Figure 1.1, left. The norm of the vectors describes the velocity of solutions; note that in Figure 1.1, left, the arrows have normalized length one.

Negative concentrations have no meaning in this model. This is reflected in the equations: solutions starting in the set $S = \{(x, y) \mid x, y \geq 0\}$ do not leave this set in the future, because the vector field at the boundary of S points inwards, cf. Figure 1.1, left. Thus, the set S is called positively invariant.

Points (x, y) where the velocity of the vector filed is zero, i.e. $f(x, y) = 0$, are called equilibria: starting at these points, one stays there for all positive times. In our example we have the three equilibria $x_0 = \left(\frac{3-\sqrt{5}}{8}, \frac{5+\sqrt{5}}{8}\right)$, $x_1 = \left(\frac{3+\sqrt{5}}{8}, \frac{5-\sqrt{5}}{8}\right)$ and $x_2 = (1, 0)$, cf. Figure 1.1. If the initial concentrations are equal to one of these equilibria, then the concentrations keep being the same. What happens, if the initial concentrations are adjacent to these equilibrium-concentrations?

If all adjacent concentrations approach the equilibrium-concentration for $t \to \infty$, then the equilibrium is called asymptotically stable. If they tend away from the equilibrium-concentration, then the equilibrium is called unstable. In the example, x_1 is unstable (grey point in Figure 1.1), while x_0 and x_2 are asymptotically stable (black points in Figure 1.1). The stability of equilibria can often be checked by linearization, i.e. by studying the Jacobian matrix $Df(x_0)$. We know that solutions with initial concentrations near the asymptotically stable equilibrium x_0 tend to x_0. But what does "near" mean?

The set of all initial conditions such that solutions tend to the equilibrium x_0 for $t \to \infty$ is called the basin of attraction $A(x_0)$ of x_0. We are interested in the determination of the basin of attraction. In our example $A(x_0)$ describes the set of initial concentrations so that the concentrations of nutrient and bacteria tend to x_0, which implies that the concentration of the bacteria tends to a constant positive value. If, however, our initial concentrations are in $A(x_2)$, then solutions tend to x_2, i.e. the bacteria will eventually die out.

The determination of the basin of attraction is achieved by a Lyapunov function. A Lyapunov function is a scalar-valued function which decreases along solutions of the differential equation. This can be verified by checking that the orbital derivative, i.e. the derivative along solutions, is negative. One can imagine the Lyapunov function to be a height function, such that solutions move downwards, cf. Figure 1.2. The Lyapunov function enables us to determine a subset K of the basin of attraction by its sublevel sets. These sublevel sets are also positively invariant, i.e. solutions do not leave them in positive time.

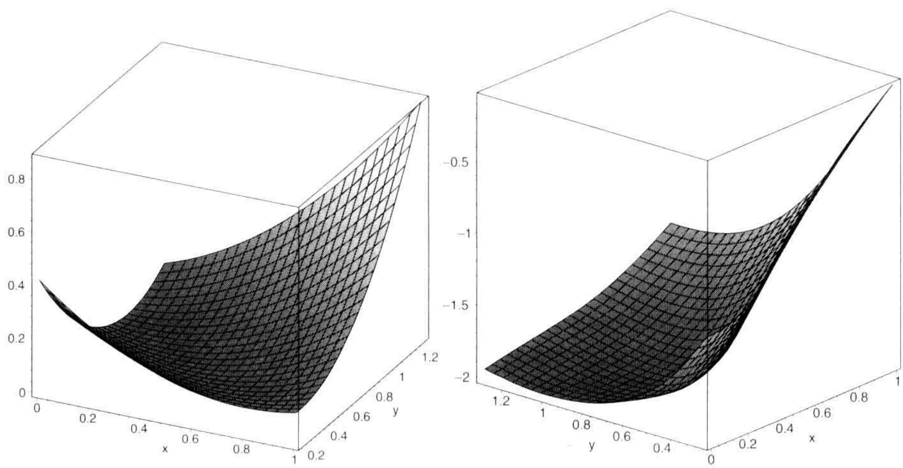

Fig. 1.2. Left: a plot of the local Lyapunov function $\mathfrak{v}$. Note that $\mathfrak{v}$ is a quadratic form. Right: A plot of the calculated Lyapunov function v.

Unfortunately, there is no general construction method for Lyapunov functions. Locally, i.e. in a neighborhood of the equilibrium, a local Lyapunov function can be calculated using the linearization of the vector field f. The orbital derivative of this local Lyapunov function, however, is only negative in a small neighborhood of the origin in general. Figure 1.2, left, shows the local Lyapunov function $\mathfrak{v}(x) = (x - x_0)^T \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5}\sqrt{5} - \frac{11}{10} \end{pmatrix} (x - x_0)$, for the determination of $\mathfrak{v}$ cf. Section 2.2.2. In Figure 1.1, right, we see that the orbital

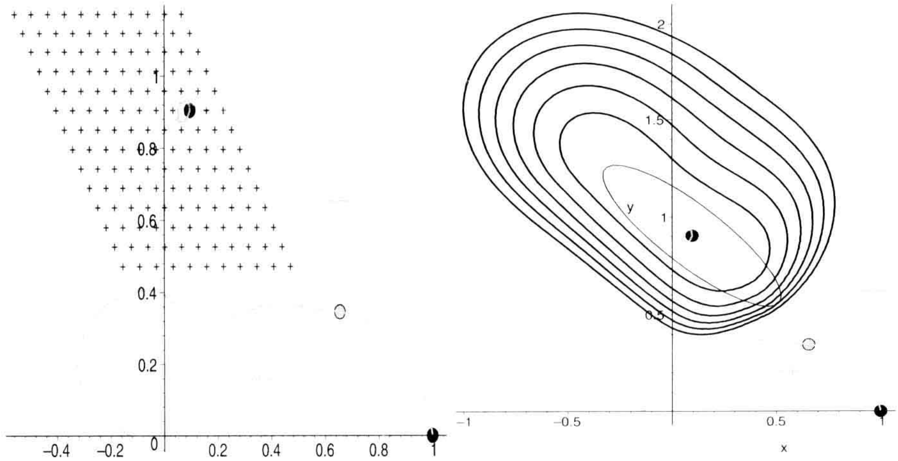

Fig. 1.3. Left: the 153 grid points (black +) for the approximation using radial basis functions and the sign of the orbital derivative $v'(x, y)$ (grey), where v is the calculated Lyapunov function using radial basis functions. Right: the sign of the orbital derivative $v'(x, y)$ (grey), level sets $v(x, y) = -1.7, -1.75, -1.8, -1.85, -1.9, -1.95$, where v is the calculated Lyapunov function using radial basis functions (black), and the sublevel set $\mathfrak{v}(x, y) \leq 0.025$ of the local Lyapunov function (thin black ellipse). This sublevel set covers the points where $v'(x, y) \geq 0$. Hence, sublevel sets of the calculated Lyapunov function are subsets of the basin of attraction $A(x_0)$.

derivative $\mathfrak{v}'$ is negative near x_0 (grey) and thus sublevel sets of $\mathfrak{v}$ (black) are subsets of the basin of attraction.

In this book we will present a method to construct a Lyapunov function in order to determine larger subsets of the basin of attraction. Figure 1.2, right, shows such a calculated Lyapunov function v. In Figure 1.3, right, we see the sign of the orbital derivative $v'(x)$ and several sublevel sets of v. Figure 1.4, left, compares the largest sublevel sets of the local and the calculated Lyapunov function.

The idea of the method evolves from a particular Lyapunov function. Although the explicit construction of a Lyapunov function is difficult, there are theorems available which prove their existence. These converse theorems use the solution of the differential equation to construct a Lyapunov function and since the solutions are not known in general, these methods do not serve to explicitly calculate a Lyapunov function. However, they play an important role for our method.

We study Lyapunov functions fulfilling equations for their orbital derivatives, e.g. the Lyapunov function V satisfying $V'(x) = -\|x - x_0\|^2$. Here, V' denotes the orbital derivative, which is given by $V'(x) = \sum_{i=1}^{2} f_i(x) \frac{\partial V}{\partial x_i}(x)$. Hence, V is the solution of a first-order partial differential equation. We approximate the solution V using radial basis functions and obtain the

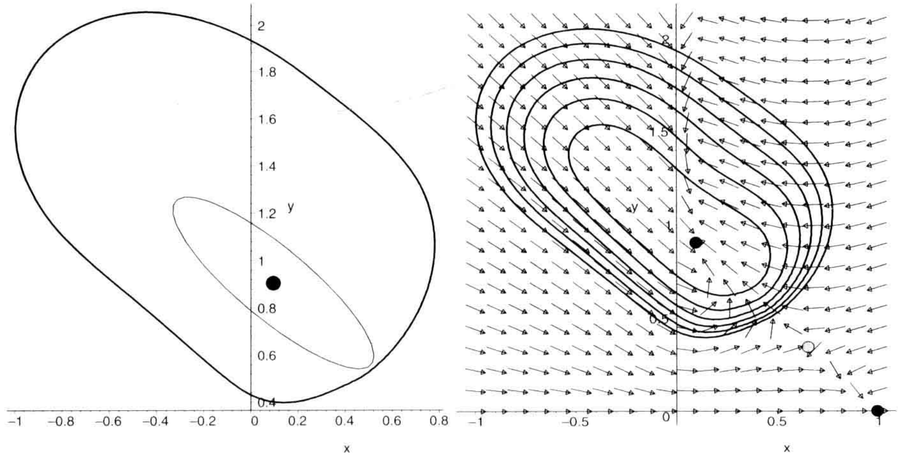

Fig. 1.4. Left: comparison of two subsets of the basin of attraction obtained by the local Lyapunov function (thin black) and by the calculated Lyapunov function (black). Right: the vector field and subsets of the basin of attraction obtained by the calculated Lyapunov function. The subsets are positively invariant – the vector field points inwards.

approximation v. Error estimates for the orbital derivative ensure $v'(x) < 0$ and, thus, the approximation v is a Lyapunov function.

For the radial basis function approximation, we fix a radial basis function $\Psi(x)$ and a grid of scattered points $X_N = \{x_1, \ldots, x_N\}$. In this book we choose the Wendland function family $\psi_{l,k}$ to define the radial basis function by $\Psi(x) = \psi_{l,k}(c\|x\|)$. We use a certain ansatz for the approximating function v and choose the coefficients such that v satisfies the partial differential equation $v'(x_j) = -\|x_j - x_0\|^2$ for all points $x_j \in X_N$ of the grid.

Figure 1.3, left, shows the grid points (black +) that were used for the calculation of v. The sign of v' is negative at the grid points because of the ansatz and also between them due to the error estimate. However, $v'(x)$ is positive near the equilibrium x_0, but this area is covered by the local Lyapunov basin, cf. Figure 1.3, right. Thus, sublevel sets of v are subsets of the basin of attraction. Figure 1.4, left, shows that the calculated Lyapunov function v determines a larger subset of the basin of attraction than the local Lyapunov function $\mathfrak{v}$. All these sublevel sets are subsets of the basin of attraction and, moreover, they are positively invariant, i.e. the vector field at the level sets points inwards, cf. Figure 1.4, right.

Hence, concerning our chemostat example, we have determined subsets of the basin of attraction of x_0, cf. Figure 1.4. If the initial concentrations in the vessel lie in such a set, then solutions tend to the equilibrium x_0 and the bacteria do not die out.

For a similar example, cf. [24], where we also consider a chemostat example, but with a different non-monotone uptake function $a(x)$.

1.2 Lyapunov Functions and Radial Basis Functions

In this section we review the literature on Lyapunov functions and radial basis functions

Lyapunov Functions

The literature on Lyapunov functions is very large; for an overview cf. Hahn [34]. In 1893, Lyapunov [48] introduced his direct or second method, where he sought to obtain results concerning the stability of an equilibrium without knowing the solution of the differential equation, but by only using the differential equation itself. He used what later was called Lyapunov functions and proved that a strict Lyapunov function implies the asymptotic stability of the equilibrium. Barbašin and Krasovskiĭ [6] showed that the basin of attraction is the whole phase space if the Lyapunov function is radially unbounded. Hahn describes how a Lyapunov function can be used to obtain a subset of the basin of attraction through sublevel sets, cf. [35] pp. 108/109 and 156/157.

Converse theorems which guarantee the existence of such a Lyapunov function under certain conditions have been given by many authors, for an overview cf. [35] or [58]. The first main converse theorem for asymptotic stability was given by Massera [50] in 1949 and it was improved by many authors in several directions. However, all the existence theorems offer no method to explicitly construct Lyapunov functions.

Krasovskiĭ writes in 1959: "One could hope that a method for proving the existence of a Lyapunov function might carry with it a constructive method for obtaining this function. This hope has not been realized", [46], pp. 11/12. He suggests [46], p. 11, to start from a given system, find a simpler system which approximates the original one and for which one can show stability, and then to prove that the corresponding property also holds for the original system.

For linear systems one can construct a quadratic Lyapunov function of the form $\mathfrak{v}(x) = (x - x_0)^T B(x - x_0)$ with a symmetric, positive definite matrix B, where x_0 denotes the equilibrium, cf. e.g. [33]. In [34], pp. 29/30, Hahn describes, starting from a nonlinear system, how to use the quadratic Lyapunov function of the linearized system as a Lyapunov function for the nonlinear system. He also discusses the search for a sublevel set inside the region $\mathfrak{v}'(x) < 0$, which is a subset of the basin of attraction.

Many approaches consider special Lyapunov functions like quadratic, polynomial, piecewise linear, piecewise quadratic or polyhedral ones, which are special piecewise linear functions. Often these methods can only be applied to special differential equations.

Piecewise linear functions are particularly appropriate for the implementation on computers since they only depend on a finite number of values. Julián [42] approximated the differential equation by a piecewise linear right-hand side and constructed a piecewise linear Lyapunov function using linear programming (linear optimization). Hafstein (formerly Marinossón), cf. [49] or [32], improved this ansatz and constructed a piecewise linear Lyapunov function for the original nonlinear system also using linear programming. Moreover, he included an error analysis in his ansatz. On the other hand he could not guarantee that the subsets which he determines with his method cover the whole basin of attraction. In some of his examples the calculated subset is even smaller than the one obtained by the Lyapunov function for the linearized system with a sharper estimate.

A different method deals with the Zubov equation and computes a solution of this partial differential equation. Since the solution of the Zubov equation determines the whole basin of attraction, one can cover each compact subset of the basin of attraction with an approximate solution. For computational aspects, cf. e.g. Genesio et al. [19]. In a similar approach to Zubov's method, Vannelli & Vidyasagar [59] use a rational function as Lyapunov function candidate and present an algorithm to obtain a maximal Lyapunov function in the case that f is analytic.

In Camilli et al. [12], Zubov's method was extended to control problems in order to determine the robust domain of attraction. The corresponding generalized Zubov equation is a Hamilton-Jacobi-Bellmann equation. This equation has a viscosity solution which can be approximated using standard techniques after regularization at the equilibrium, e.g. one can use piecewise affine approximating functions and adaptive grid techniques, cf. Grüne [30] or Camilli et al. [12]. The method works also for non-smooth f since the solution is not necessarily smooth either. The error estimate here is given in terms of $|v_\epsilon(x) - \tilde{v}_\epsilon(x)|$, where v_ϵ denotes the regularized Lyapunov function and $\tilde{v}_\epsilon$ its approximation, and not in terms of the orbital derivative.

In this book we present a new method to construct Lyapunov functions. We start from a converse theorem proving the existence of a Lyapunov function T which satisfies the equation $T'(x) = -\bar{c}$, where $\bar{c} > 0$ is a given constant. This equation is a linear first-order partial differential equation due to the formula for the orbital derivative:

$$T'(x) = \sum_{i=0}^{n} f_i(x) \frac{\partial T}{\partial x_i}(x) = -\bar{c}. \tag{1.2}$$

The main goal of this book is to approximate the solution T of (1.2) by a function t using radial basis functions. It turns out that t itself is a Lyapunov function, i.e. $t'(x)$ is negative, and thus can be used to determine the basin of attraction. The approximation error will be estimated in terms of $|T'(x) - t'(x)| \leq \iota$. Hence, $t'(x) \leq T'(x) + \iota = -\bar{c} + \iota < 0$ if the error $\iota < \bar{c}$ is small enough.

However, the function T is not defined at x_0. Hence, we consider a second class of Lyapunov functions V which are defined and smooth at x_0. They satisfy the equation $V'(x) = -p(x)$, where $p(x)$ is a given function with certain properties, in particular $p(x_0) = 0$, and we often use $p(x) = \|x - x_0\|^2$. The equation $V'(x) = -p(x)$ is a modified Zubov equation, cf. [24]. In the following we denote by Q one of these Lyapunov function of type $Q = T$ or $Q = V$. For the approximation of Q we use radial basis functions.

Radial Basis Functions

Radial basis functions are a powerful tool to solve partial differential equations. For an overview cf. [63], [11], [10], or [52], for a tutorial cf. [40]. The main advantage of this method is that it is meshless, i.e. no triangulation of the space $\mathbb{R}^n$ is needed. Other methods, e.g. finite element methods, first generate a triangulation of the space, use functions on each part of the triangulation, e.g. affine functions as in some examples discussed above, and then patch them together obtaining a global function. The resulting function is not very smooth and the method is not very effective in higher space dimensions. Moreover, the interpolation problem stated by radial basis functions is always uniquely solvable. Radial basis functions give smooth approximations, but at the same time require smooth functions that are approximated. In our applications, as we will see, the freedom of choosing the grid in an almost arbitrary way will be very advantageous.

Let us explain the approximation with radial basis functions. Denote by D the linear operator of the orbital derivative, i.e. $DQ(x) = Q'(x)$. We use the symmetric ansatz leading to a symmetric interpolation matrix A. One defines a grid $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$. The reconstruction (approximation) q of the function Q is obtained by the ansatz $q(x) = \sum_{k=1}^{N} \beta_k \langle \nabla_y \Psi(x-y)\big|_{y=x_k}, f(x_k)\rangle$ with coefficients $\beta_k \in \mathbb{R}$. The function Ψ is the radial basis function. In this book we use $\Psi(x) = \psi_{l,k}(c\|x\|)$ where $\psi_{l,k}$ is a Wendland function. Wendland functions are positive definite functions (and not only conditionally positive definite) and have compact support. The coefficients β_k are determined by the claim that $q'(x_j) = Q'(x_j)$ holds for all grid points $j = 1, \ldots, N$. This is equivalent to a system of linear equations $A\beta = \alpha$ where the interpolation matrix A and the right-hand side vector α are determined by the grid and the values $Q'(x_j)$. The interpolation matrix A is a symmetric $(N \times N)$ matrix, where N is the number of grid points. We show that A is positive definite and thus the linear equation has a unique solution β. Provided that Q is smooth enough, one obtains an error estimate on $|Q'(x) - q'(x)|$ depending on the density of the grid.

While the interpolation of function values has been studied in detail since the 1970s, the interpolation via the values of a linear operator and thus the solutions of PDEs has only been considered since the 1990s. The values of such linear operators are also called Hermite-Birkhoff data. They have been studied, e.g. by Iske [38], Wu [67], Franke & Schaback [17] and [18] and Wendland [63]. Franke & Schaback approximated the solution of a Cauchy problem in partial differential equations, cf. also [54]. This results in a mixed problem, combining different linear operators, cf. [17] and [18]. Their error estimates used the fact that the linear operator is translation invariant. The partial differential equations they studied thus have constant coefficients. Our linear