Applied Analytical Mathematics for Physical Scientists

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Preface

The selection of material and the form of presentation for this book grew out of several courses in applied mathematics that I have taught to graduate and undergraduate students of physics, chemistry, and engineering over the past 10 years. Two main problems face an instructor of applied mathematics courses that are intended for advanced students in the physical sciences. Since there are more topics of importance than can be covered in a typical one-year course, a choice of subjects must be made, based on the needs of the students and the prejudices of the instructor. Second, the style of presentation of the topics can range from the cookbook method of developing many results and tricks by studying physical applications to the completely general and rigorous abstract approach.

It is for these two reasons that this textbook on applied mathematics now competes for the attention of physical science students. I have been unable to find a single textbook that covered all of the necessary topics and treated them with sufficient rigor and generality while still remaining intelligible to most students in the class. This is a value judgment that cannot be defended in absolute terms. The fact that many other texts exist in this general area shows that several competent authors have reached a conclusion very different from mine. This in itself is not criticism of their work nor, I hope, of mine. Readers will choose the approach that best fits their needs.

The purpose of this text is to cover the topics that are necessary to give the student a sufficiently broad background for his advanced studies in the physical sciences and to present this material with enough rigor and generality to provide him with a unified view and a solid understanding of this material. The philosophy underlying the development of this text is that it is as important for a student to know *when* a given theorem or result applies and to be aware of the subtlety of certain questions as it for him to be familiar with the mechanics of the applications of these principles. One of the most important goals to be achieved in studying mathematics is the development of a certain method of thinking and a style of approach to a problem. Nevertheless, technical skills must be developed and there are many problems included to accomplish this, as well as to extend the material of the text.

I assume that students have a calculus background through the differentiation and integration of functions of several real variables. On this assumption I have usually been able to cover Chapters 1 through 6 the first semester and Chapters

7 through 9 the second. It is generally true that physical science students have strengths in certain areas of mathematics and weaknesses in others as a result of the use they have made of mathematics in their physical sciences courses. The text attempts to build on these strengths and not to cover old material unnecessarily. It does not abound with "typical" hackneyed examples from physics and engineering, since students taking this course invariably have had or are taking courses in mechanics, electricity and magnetism, quantum mechanics, or other such courses that apply many of the techniques developed here. No attempt is made to provide direct motivation for each topic covered by illustrating these techniques with specific applications. The student should have enough sophistication to study the mathematics itself. It has been my experience that students do not enjoy or greatly profit from going over the same applications in several different courses. Such specific problems receive much more detailed consideration in other advanced courses. The most general and abstract discussions and proofs are not always given but, instead, those that the student's background will allow him to understand well. Nevertheless, the assumptions made in proving important theorems are kept as weak as is consistent with this goal. Many details of proofs and some advanced topics have been relegated to chapter appendices to facilitate the development of central ideas in each chapter. Useful results are listed in tables at the end of various chapters for ease of reference.

Since most physical science students for whom this text is designed have a working familiarity with, or little difficulty in grasping, three-dimensional vector analysis and even the elements of tensor analysis, no time is spent on these topics, although frequent reference is made to simple analogies from these fields. No knowledge of the Lebesgue theory of integration is assumed, even though it is only with this that concepts such as completeness and the theory of linear operators on a Hilbert space can be fully appreciated. At several points in the first chapter, and later in the text, the symbol $\mathscr{L}_2$ (standing for Lebesgue square-integrable) appears. The reader can usually grasp the essence of the point being made by thinking in terms of the more familiar Riemann definition of an integral. A brief discussion of measure and of the Lebesgue definition of an integral is given in Appendix II. Symbolic notation has been used often in the text since it sometimes makes an argument sufficiently compact that is more easily understood. Furthermore, the student will find reference to more advanced works easier once he has become accustomed to it.

Clearly the most important unifying concepts in the text are those of a linear vector space and of the theory of linear operators on such spaces. Chapter 2 sets the style for most of the further considerations of the book. The debt owed for this central chapter, and for much of the text, goes to Bernard Friedman's *Principles and Techniques of Applied Mathematics*. In fact it is my belief that, had the late Professor Friedman chosen to write another book covering a few more topics and with more background material, but with the same beautiful style of clarity and logic as his *Principles and Techniques*, then many of the applied mathematics texts published in the last 15 years would have been unnecessary. Chapter 2 also lays the foundation for the work on integral equations in the discussion of completely continuous linear operators. Chapter 3 develops many important properties of linear operators, mainly in finite-dimensional Hilbert spaces.

Chapter 4 deals with the very important topic of complete sets of functions in terms of the Weierstrass approximation theorem and the basic theorem on Fourier series. Other complete sets of functions, such as Legendre polynomials, are also introduced. Separation of variables and the subsequent applications to boundary value problems are not given any great space either here or later in the text. Most students already know how to apply the method but have little understanding of the concept of completeness, which is discussed extensively here. Rigorous treatments of Fourier series and of Fourier integrals are given in as general a form as is possible without using results from Lebesgue integration theory.

The principal result on the existence and uniqueness of the solution to Volterra integral equations, established early in Chapter 5, serves as a basis for a later discussion of ordinary differential equations. The main business of Chapter 5 is a development of Fredholm theory culminating in a proof of the Hilbert-Schmidt theorem. The basic approach taken for several topics in this chapter is that of the limit as $n \to \infty$ of a set of linear algebraic equations in a representation-dependent framework. Although this is somewhat cumbersome in detail for a few of the proofs, it is very easily understood conceptually. The more elegant and powerful technique of defining completely continuous operators in terms of sequences and convergent subsequences of vectors is also introduced in the text and developed in an appendix.

Chapter 6 is a brief introduction to the calculus of variations and follows the basic approach of the classic small volume by Gilbert A. Bliss. There is also a discussion of Noether's theorem that plays such an important role in modern formulations of mechanics and field theories.

The long Chapter 7 on complex variables stresses methods of analytic continuation, especially as applied to the classical functions of mathematics. Very little is done with conformal mapping, since this is a rather specialized tool presenting no conceptual difficulty but best learned in some specific area of application. Complex variables is a very useful subject for physical scientists, who seem to have relatively little difficulty with it, unlike discussions of sets, completeness, and Hilbert spaces, for example. Perhaps this is easily understood since Riemann formulated many of his great theorems in complex variable theory by considering the idealized flow of electric charges along thin conducting surfaces. Illustrations prove very useful here and are used abundantly.

Chapter 8, on linear differential equations and Green's functions, begins with a brief introduction to distributions or ideal functions. A fairly complete and rigorous treatment is then given of Green's functions for second-order linear differential equations in a single variable. A discussion of the important topic of the continuous spectrum for differential operators in an infinite domain is given. Finally Green's functions are developed for some common differential operators in three dimensions, but with considerably less rigor than in the one-dimensional case.

Chapter 9 on group theory has been included because this is a subject of considerable importance to physical scientists today, and it is one that they are very often unfamiliar with. After a general introduction most of the emphasis is placed on continuuus Lie groups and in particular on the three-dimensional rotation group and its representations. Eugene P. Wigner's classic text, first published in 1931, serves as the basic reference, as well as Giulio Racah's famous lectures on continuous groups. The latter half of this chapter is biased in favor of those students of chemistry or physics who will study quantum mechanics.

Appendix I is a convenient reference for some useful results from that area of real analysis often referred to as advanced calculus. Appendix II contains an extremely sketchy discussion of the concept of measure and of the Lebesgue definition of integration, as well as a collection of several important theorems from functional analysis for the reader who is interested in seeing how some of the results given in the text, especially those of Chapters 4 and 5, can be generalized. However these advanced theorems are never used in the text itself.

Naturally, very little new will be found in a text of this type covering these topics in applied mathematics for the beginning graduate or advanced undergraduate student. Each subject covered is treated more extensively in works devoted to that subject alone. This book provides the student with a single, unified, understandable presentation of the material.

Unfortunately, when a text is developed from so many different sources over a period of several years, it is not always possible to recall exactly the sources used and to quote them. I have tried to indicate my major references at the end of each chapter and in the bibliography. Nearly four hundred years ago Sir Francis Bacon in his *Nuovum Organum* charged: "... let a man look carefully into all that variety of books with which the arts and sciences abound, he will find everywhere endless repetitions of the same thing, varying in the method of treatment, but not new in substance ...." Indeed it is often so in the mathematical literature that the original proofs and discussions given by the great classical mathematicians are so clear and beautiful and that subsequent works have essentially copied them. This is not a criticism of lesser authors since it would serve no purpose to concoct an original proof at the expense of preciseness or clarity. The present text is not free from such charges of nonoriginality in several key discussions.

I have tried to keep my personal prejudices as a theoretical physicist from coming through too blatantly in the selection and presentation of subject matter, although it is clear that I have yielded to temptation on several occasions, especially in the last sections of Chapter 9.

Finally I thank Brian Cheng-jean Chen, Gerald L. Jones, and William D. McGlinn for discussions of various points during the writing of this text and my students, who have endured the preliminary versions of the notes used for this course, for their questions and suggestions. I also thank Sharon Duram and Eleanor Klingbeil, who typed the bulk of the lengthy first draft, and Jo Robertson, who typed and retyped the extensive revisions. Naturally, those errors that remain are mine, and I welcome any comments on, corrections to, or criticisms of this text.

JUNE, 1974 JAMES T. CUSHING

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1 Linear Vector Spaces

1.0 INTRODUCTION

As we will repeatedly stress throughout the text, many of the concepts we will use are simply generalizations of those already encountered in ordinary three-dimensional vector analysis. Consider a three-vector $\mathbf{x}=(x_1, x_2, x_3)$, where x_1, x_2, x_3 are the components of $\mathbf{x}$ along three mutually orthogonal Cartesian axes. We say that $\mathbf{x}$ is a vector in E_3 (i.e., three-dimensional Euclidean space). Suppose we wish to find an $\mathbf{x}$ such that

$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3&=b_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3&=b_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3&=b_3 \end{aligned} \tag{1.1}$$

Such systems of inhomogeneous linear equations are studied in elementary algebra.

Let us state some results for these systems by way of review. The reader who is unfamiliar with the following results will find proofs outlined in Problem 1.1 at the end of this chapter. If we define a quantity Δ as

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\equiv a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22}) \tag{1.2}$$