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Ali Süleyman Üstünel

An Introduction to Analysis on Wiener Space

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Introduction

The following pages are the notes from a seminar that I gave during the spring and some portion of the summer of 1993 at the Mathematics Institute of Oslo University. The aim of the seminars was to give a rapid but rigorous introduction for the graduate students to Analysis on Wiener space, a subject which has grown up very quickly these recent years under the impulse of the Stochastic Calculus of Variations of Paul Malliavin (cf. [12]).

Although some concepts are in the preliminaries, I assumed that the students had already acquired the notions of stochastic calculus with semimartingales, Brownian motion and some rudiments of the theory of Markov processes. A small portion of the material exposed is our own research, in particular, with Moshe Zakai. The rest has been taken from the works listed in the bibliography.

The first chapter deals with the definition of the (so-called) Gross-Sobolev derivative and the Ornstein-Uhlenbeck operator which are indispensable tools of the analysis on Wiener space. In the second chapter we begin the proof of the Meyer inequalities, for which the hypercontractivity property of the Ornstein-Uhlenbeck semigroup is needed. We expose this last topic in the third chapter, then come back to Meyer inequalities, and complete their proof in chapter IV. Different applications are given in next two chapters. In the seventh chapter we study the independence of some Wiener functionals with the previously developed tools. The chapter VIII is devoted to some series of moment inequalities which are important for applications like large deviations, stochastic differential equations, etc. In the last chapter we expose the contractive version of Ramer's theorem as another example of the applications of moment inequalities developed in the preceding chapter.

During my visit to Oslo, I had the chance of having an ideal environment for working and a very attentive audience in the seminars. These notes have particularly profited from the serious criticism of my colleagues and friends Bernt Øksendal, Tom Lindstrøm, Ya-Zhong Hu, and the graduate students of the Mathematics department. It remains for me to express my gratitude also to Nina Haraldsson for her careful typing, and, last but not least, to Laurent Decreusefond for correcting so many errors .

Ali Süleyman Üstünel

Contents

Preliminaries

This chapter is devoted to the basic results about the Wiener measure, Brownian motion, construction of the Ito stochastic integral and the chaos decomposition associated to it.

1 The Brownian Motion and the Wiener Measure

1) Let $W = C_0([0,1])$, $\omega \in W$, $t \in [0,1]$, define $W_t(\omega) = \omega(t)$ (the coordinate functional). If we note by $\mathcal{B}_t = \sigma\{W_s; s \leq t\}$, then there is one and only one measure μ on W such that

i) $\mu\{W_0(\omega) = 0\} = 1$,

ii) $\forall f \in C_b^\infty(\mathbf{R})$, the stochastic process process

$$(t,\omega) \mapsto f(W_t(\omega)) - \frac{1}{2}\int_0^t f''(W_s(\omega))ds$$

is a $(\mathcal{B}_t, \mu)$-martingale. μ is called the Wiener measure.

2) From the construction we see that for $t > s$,

$$E_\mu[\exp i\alpha(W_t - W_s)|\mathcal{B}_s] = \exp -\alpha^2(t-s),$$

hence $(t,\omega) \mapsto W_t(\omega)$ is a continuous additive process (i.e.,a process with independent increments) and $(W_t; t \in [0,1])$ is also a continous martingale.

3) Stochastic Integration

Let $K : W \times [0,1] \to \mathbf{R}$ be a step process :

$$K_t(\omega) = \sum_{i=1}^n a_i(\omega) \cdot 1_{[t_i,t_{i+1}[}(t), \qquad a_i(\omega) \in L^2(\mathcal{B}_{t_i}).$$

Define

$$I(K) = \int_0^1 K_s dW_s(\omega)$$

as

$$\sum_{i=1}^{n} a_i(\omega)\cdot(W_{t_{i+1}}(\omega)-W_{t_i}(\omega)).$$

Then we have

$$E\left[\left(\int_0^1 K_s dW_s\right)^2\right]=E\int_0^1 K_s^2 ds,$$

i.e. I is an isometry from the adapted step processes into $L^2(\mu)$, hence it has a unique extension as an isometry from

$$L^2([0,1]\times W,\mathcal{A},dt\times d\mu)\xrightarrow{I}L^2(\mu)$$

where $\mathcal{A}$ denotes the sigma algebra on $[0,1]\times W$ generated by the adapted, left (or right) continuous processes. $I(K)$ is called the stochastic integral of K and it is denoted as $\int_0^1 K_s dW_s$. If we define

$$I_t(K)=\int_0^t K_s dW_s$$

as

$$\int_0^1 1_{[0,t]}(s)K_s dW_s,$$

it is easy to see that the stochastic process $t\mapsto I_t(K)$ is a continuous, square integrable martingale. With some localization techniques using stopping times, I can be extended to any adapted process K such that $\int_0^1 K_s^2(\omega)ds<\infty$ a.s. In this case the process $t\mapsto I_t(K)$ becomes a local martingale, i.e., there exists a sequence of stopping times increasing to one, say $(T_n, n\in\mathbf{N})$ such that the process $t\mapsto I_{t\wedge T_n}(K)$ is a (square integrable) martingale.

Application: Ito formula We have following important applications of the stochastic integration:

a) If $f\in C^2(\mathbf{R})$ and $M_t=\int_0^t K_r dW_r$, then

$$f(M_t)=f(0)+\int_0^t f'(M_s)K_s dW_s+\frac{1}{2}\int_0^t f''(M_s)K_s^2 ds.$$

b)

$$\mathcal{E}_t(I(h))=\exp\left(\int_0^t h_s dW_s-\frac{1}{2}\int_0^t h_s^2 ds\right)$$

is a martingale for any $h\in L^2[0,1]$.

4) Alternative constructions of the Brownian motion and the Wiener measure

A) Let $(\gamma_i; i \in \mathbf{N})$ be an independent sequence of $N_1(0,1)$ Gaussian random variables. Let (g_i) be a complete, orthonormal basis of $L^2[0,1]$. Then W_t defined by

$$W_t(\omega) = \sum_{i=1}^{\infty} \gamma_i(\omega) \cdot \int_0^t g_i(s)ds$$

is a Brownian motion.

Remark: If $(g_i; i \in \mathbf{N})$ is a complete, orthonormal basis of $L^2([0,1])$, then $\left(\int_0^{\cdot} g_i(s)ds; i \in \mathbf{N}\right)$ is a complete orthonormal basis of $H([0,1])$ (i. e., the first order Sobolev functionals on $[0,1]$).

B) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be any abstract probability space and let H be any separable Hilbert space. If $L : H \to L^2(\Omega, \mathcal{F}, \mathbf{P})$ is a linear operator such that for any $h \in H$, $E[\exp iL(h)] = \exp -\frac{1}{2}|h|_H^2$, then there exists a Banach space with dense injection $H \overset{j}{\hookrightarrow} W$ dense, hence $W^* \overset{j^*}{\hookrightarrow} H$ is also dense and a probability measure μ on W such that

$$\int \exp\langle \omega^*, \omega\rangle d\mu(\omega) = \exp -\tfrac{1}{2} \mid j^*(\omega^*) \mid_H^2$$

and

$$L(j^*(\omega^*))(\omega) = \langle \omega^*, \omega \rangle$$

almost surely. (W, H, μ) is called an Abstract Wiener space and μ is the Wiener measure. If $H([0,1]) = \{h : h(t) = \int_0^t \dot{h}(s)ds, |h|_H = |\dot{h}|_{L^2[0,1]}\}$ then μ is the classical Wiener measure and W can be taken as $C_0([0,1])$.

Remark: In the case of the classical Wiener space, any element λ of W^* is a signed measure on $[0,1]$, and its image in $H = H([0,1])$ can be represented as $j^*(\lambda)(t) = \int_0^t \lambda([s,1])ds$. In fact, we have for any $h \in H$

$$\begin{aligned}
(j^*(\lambda), h) &= <\lambda, j(h)> \\
&= \int_0^1 h(s)\lambda(ds) \\
&= h(1)\lambda([0,1]) - \int_0^1 \lambda([0,s])\dot{h}(s)ds \\
&= \int_0^1 (\lambda([0,1]) - \lambda([0,s])\dot{h}(s)ds \\
&= \int_0^1 \lambda([s,1])\dot{h}(s)ds.
\end{aligned}$$

5) Let us come back to the classical Wiener space:

i) It follows from the martingale convergence theorem and the monotone class theorem that the set of random variables

$$\{f(W_{t_1},\ldots,W_{t_n}); t_i \in [0,1], f \in \mathcal{S}(\mathbf{R}^n); n \in \mathbf{N}\}$$

is dense in $L^2(\mu)$, where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of infinitely differentiable, rapidly decreasing functions on $\mathbf{R}^n$.

ii) It follows from (i), via the Fourier transform that the linear span of the set $\{\exp \int_0^1 h_s dW_s - \frac{1}{2}\int_0^1 h_s^2 ds; h \in L^2([0,1])\}$ is dense in $L^2(\mu)$.

iii) Because of the analyticity of the characteristic function of the Wiener measure, the elements of the set in (ii) can be approached by the polynomials, hence the polynomials are dense in $L^2(\mu)$.

5.1 Cameron-Martin Theorem:

For any bounded Borel measurable function F, $h \in L^2[0,1]$, we have

$$E_\mu[F(w + \int_0^\bullet h_s ds) \cdot \exp[-\int_0^1 h_s dW_s - \frac{1}{2}\int_0^1 h_s^2 ds]] = E_\mu[F].$$

This means that the process $W_t(\omega) + \int_0^t h_s ds$ is again a Brownian motion under the new probability measure

$$\exp(-\int_0^1 h_s dW_s - \frac{1}{2}\int_0^1 h_s^2 ds) d\mu.$$

Proof: It is sufficient to show that the new probability has the same characteristic function as μ: if $x^* \in W^*$, then x^* is a measure on $[0,1]$ and

$$\begin{aligned} {}_{W^*}\langle x^*, w\rangle_W &= \int_0^1 W_s(\omega) x^*(ds) \\ &= W_t(\omega)\cdot x^*([0,t])\Big|_0^1 - \int_0^1 x^*([0,t]) dW_t(\omega) \\ &= W_1 x^*([0,1]) - \int_0^1 x^*([0,t]).dW_t \\ &= \int_0^1 x^*(]t,1]) dW_t\,. \end{aligned}$$

Consequently

$$
\begin{aligned}
& E[\exp i \int_0^1 x^*([t,1])dW_t(w + \int_0^\bullet h_s ds) \cdot \mathcal{E}(-I(h))] \\
= \quad & E[\exp i \int_0^1 x^*([t,1])dW_t + i \int_0^1 x^*([t,1])h_t dt - \int_0^1 h_t dW_t - \frac{1}{2}\int_0^1 h_t^2 dt] \\
= \quad & E[\exp i \int_0^1 (ix^*([t,1]) - h_t)dW_t . \exp i \int_0^1 x^*([t,1])h_t dt - \frac{1}{2}\int_0^1 h_t^2 dt] \\
= \quad & \exp \frac{1}{2}\int_0^1 (ix^*([t,1]) - h_t)^2 dt + i \int_0^1 x^*([t,1])h_t dt - \frac{1}{2}\int_0^1 h_t^2 dt \\
= \quad & \exp -\frac{1}{2}\int_0^1 (x^*([t,1]))^2 dt \\
= \quad & \exp -\frac{1}{2} \mid j(x^*) \mid_H^2 .
\end{aligned}
$$

QED

Corollary (Paul Lévy's Theorem) Suppose that (M_t) is a continuous martingale such that $M_0 = 0$, $M_t^2 - t$ is again a martingale. Then (M_t) is a Brownian motion.

Proof: We have the Ito formula

$$f(M_t) = f(0) + \int_0^t f'(M_s) \cdot dM_s + \frac{1}{2}\int_0^t f''(M_s) \cdot ds .$$

Hence the law of $\{M_t : t \in [0,1]\}$ is μ. QED

5.2 The Ito Representation Theorem:

Any $\varphi \in L^2(\mu)$ can be represented as

$$\varphi = E[\varphi] + \int_0^1 K_s dW_s$$

where $K \in L^2([0,1] \times W)$, adapted.

Proof: Since the Wick exponentials

$$\mathcal{E}(I(h)) = \exp \int_0^1 h_s dW_s - 1/2 \int_0^1 h_s^2 ds$$

can be represented as claimed, the proof follows by density. QED

5.3 Wiener chaos representation

Let $K_1 = \int_0^1 h_s dW_s$, $h \in L^2([0,1])$. Then, from the Ito formula, we can write

$$K_1^p = p\int_0^1 K_s^{p-1} h_s dW_s + \frac{p(p-1)}{2}\int_0^1 K_s^{p-2} h_s^2 ds$$
$$= p\int_0^1 \left[(p-1)\int_0^{t_1} K_{t_2}^{p-2} h_{t_2} dW_{t_2} + \frac{(p-1)(p-1)}{2}\int_0^{t_1} K_{t_2}^{p-3} h_{t_2}^2 dt_2\right] dW_{t_1}$$
$$+ \cdots$$

iterating this procedure we end up on one hand with $K_{t_p}^0 = 1$, on the other hand with the multiple integrals of deterministic integrands of the type

$$J_p = \int_{0<t_p<t_{p-1}<\cdots\leq t_1<1} h_{t_1} h_{t_2} \ldots h_{t_p}\, dW_{t_1}^{i_1} \ldots dW_{t_p}^{i_p},$$

$i_j = 0$ or 1 with $dW_t^0 = dt$ and $dW_t^1 = dW_t$.

Let now $\varphi \in L^2(\mu)$, then we have from the Ito representation theorem

$$\varphi = E[\varphi] + \int_0^1 K_s dW_s$$

by iterating the same procedure for the integrand of the above stochastic integral:

$$\varphi = E[\varphi] + \int_0^1 E[K_s] dW_s + \int_0^1\int_0^{t_1} E[K_{t_1,t_2}^{1,2}] dW_{t_2} dW_{t_1} +$$
$$+ \int_0^1\int_0^{t_1}\int_0^{t_2} K_{t_1 t_2 t_3}^{1,2,3} dW_{t_3} dW_{t_2} dW_{t_1}\,.$$

After N iterations we end up with

$$\varphi = \sum_0^N J_p(K^p) + \varphi_{N+1}$$

and each element of the sum is orthogonal to the other one. Hence $(\varphi_N; N \in \mathbf{N})$ is bounded in $L^2(\mu)$. Let (φ_{N_k}) be a weakly convergent subsequence and $\varphi_\infty = \lim_{k\to\infty} \varphi_{N_k}$. Then it is easy from the first part that φ_∞ is orthogonal to the polynomials, therefore $\varphi_\infty = 0$ and $w - \lim_{N\to\infty} \sum_0^N J_p(K_p)$ exists, moreover $\sup_N \sum_1^N \|J_p(K_p)\|_2^2 < \infty$, hence $\sum_1^\infty J_p(K_p)$ converges in $L^2(\mu)$. Let now $\widehat{K}_p$ be an element of $\widehat{L}^2[0,1]^p$ (i.e. symmetric), defined as $\widehat{K}_p = K_p$ on $C_p = \{t_1 < \cdots < t_p\}$. We define $I_p(\widehat{K}_p) = p! J_p(K_p)$ in such a way that

$$E[|I_p(\widehat{K}_p)|^2] = (p!)^2 \int_{C_p} K_p^2 dt_1 \ldots dt_p = p! \int_{[0,1]^p} |\widehat{K}_p|^2 dt_1 \ldots dt_p\,.$$

Let $\varphi_p = \frac{\widehat{K}_p}{p!}$, then we have

$$\boxed{\varphi = E[\varphi] + \sum_{1}^{\infty} I_p(\varphi_p)} \qquad \text{(Wiener chaos decomposition)}$$

Chapter I

Gross-Sobolev Derivative, Divergence and Ornstein-Uhlenbeck Operator

Motivations

Let $W = C_0([0,1], \mathbf{R}^d)$ be the classical Wiener space equipped with μ the Wiener measure. We want to construct on W a Sobolev type analysis in such a way that we can apply it to the random variables that we encounter in the applications. Mainly we want to construct a differentiation operator and to be able to apply it to practical examples. The Fréchet derivative is not satisfactory. In fact the most frequently encountered Wiener functionals, as the multiple (or single) Wiener integrals or the solutions of stochastic differential equations with smooth coefficients are not even continuous with respect to the Fréchet norm of the Wiener space. Therefore, what we need is in fact to define a derivative on the $L^p(\mu)$-spaces of random variables, but in general, to be able to do this, we need the following property which is essential: if $F, G \in L^p(\mu)$, and if we want to define their directional derivative, in the direction, say $\tilde{w} \in W$, we write $\frac{d}{dt}F(w+t\tilde{w})|_{t=0}$ and $\frac{d}{dt}G(w+t\tilde{w})|_{t=0}$. If $F = G$ μ-a.s., it is natural to ask that their derivatives are also equal a.s. For this, the only way is to choose $\tilde{w}$ in some specific subspace of W, namely, the Cameron-Martin space H:

$$H = \left\{ h : [0,1] \to \mathbf{R}^d / h(t) = \int_0^t \dot{h}(s)ds, \qquad |h|_H^2 = \int_0^1 |\dot{h}(s)|^2 ds \right\}.$$

In fact, the theorem of Cameron-Martin says that for any $F \in L^p(\mu)$, $p > 1$,

$h \in H$

$$E_\mu[F(w+h)\exp[-\int_0^1 \dot{h}(s)\cdot dW_s - \frac{1}{2}|h|_{H_1}^2]] = E_\mu[F],$$

or equivalently

$$E_\mu[F(w+h)] = E[F(w)\cdot \exp\int_0^1 \dot{h}_s \cdot dW_s - \frac{1}{2}|h|_{H_1}^2].$$

That is to say, if $F = G$ a.s., then $F(\cdot + h) = G(\cdot + h)$ a.s. for all $h \in H$.

1 The Construction of ∇ and its properties

If $F : W \to \mathbf{R}$ is a function of the following type (called cylindrical):

$$F(w) = f(W_{t_1}(w), \dots, W_{t_n}(w)), \qquad f \in \mathcal{S}(\mathbf{R}^n),$$

we define, for $h \in H$,

$$\nabla_h F(w) = \frac{d}{d\lambda} F(w + \lambda h)|_{\lambda=0}.$$

Noting that $W_t(w+h) = W_t(w) + h(t)$, we obtain

$$\nabla_h F(w) = \sum_{i=1}^n \partial_i f(W_{t_1}(w), \dots, W_{t_n}(w))h(t_i),$$

in particular

$$\nabla_h W_t(w) = h(t) = \int_0^t \dot{h}(s)ds = \int_0^1 1_{[0,t]}(s)\,\dot{h}(s)ds.$$

If we denote by U_t the element of H defined as $U_t(s) = \int_0^s 1_{[0,t]}(r)dr$, we have $\nabla_h W_t(\omega) = (U_t, h)_H$. Looking at the linear map $h \mapsto \nabla_h F(\omega)$ we see that it defines a random element with values in H, i.e. ∇F is an H-valued random variable. Now we can prove:

Prop. I.1: ∇ is a closable operator on any $L^p(\mu)$ $(p > 1)$.

Proof: This means that if $(F_n : n \in \mathbf{N})$ are cylindrical functions on W, such that $F_n \to 0$ in $L^p(\mu)$ and if $(\nabla F_n; n \in \mathbf{N})$ is Cauchy in $L^p(\mu, H)$, then its limit is zero. Hence suppose that $\nabla F_n \to \xi$ in $L^p(\mu; H)$.

To prove $\xi = 0$ μ-a.s., we use the Cameron-Martin theorem: Let φ be any cylindrical function. Since such φ's are dense in $L^p(\mu)$, it is sufficient to prove