

17

Lamberto Cesari

Optimization— Theory and Applications

Problems with Ordinary
Differential Equations



Springer-Verlag

New York Heidelberg Berlin

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Differential Equations

With 82 Illustrations



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AMS Subject Classifications: 49-02, 93-02

Library of Congress Cataloging in Publication Data

Cesari, Lamberto.
Optimization-theory and applications.
(Applications of mathematics; 17)
Bibliography: p.
Includes index.
1. Calculus of variations. 2. Mathematical
optimization. 3. Differential equations. I. Title.
II. Series.
QA316.C47 515'.64 82-5776
AACR2

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Typeset by Syntax International, Singapore.

Printed and bound by R. R. Donnelley & Sons, Harrisonburg, VA.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90676-2 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90676-2 Springer-Verlag Berlin Heidelberg New York

*Applied Probability
Control
Economics
Information and Communication
Modeling and Identification
Numerical Techniques
Optimization*

**Applications of
Mathematics**

17

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Preface

This book has grown out of lectures and courses in calculus of variations and optimization taught for many years at the University of Michigan to graduate students at various stages of their careers, and always to a mixed audience of students in mathematics and engineering. It attempts to present a balanced view of the subject, giving some emphasis to its connections with the classical theory and to a number of those problems of economics and engineering which have motivated so many of the present developments, as well as presenting aspects of the current theory, particularly value theory and existence theorems. However, the presentation of the theory is connected to and accompanied by many concrete problems of optimization, classical and modern, some more technical and some less so, some discussed in detail and some only sketched or proposed as exercises.

No single part of the subject (such as the existence theorems, or the more traditional approach based on necessary conditions and on sufficient conditions, or the more recent one based on value function theory) can give a sufficient representation of the whole subject. This holds particularly for the existence theorems, some of which have been conceived to apply to certain large classes of problems of optimization.

For all these reasons it is essential to present many examples (Chapters 3 and 6) before the existence theorems (Chapters 9 and 11–16), and to investigate these examples by means of the usual necessary conditions, sufficient conditions, and value function theory.

This book only considers nonparametric problems of the calculus of variations in one independent variable and problems of optimal control monitored by ordinary differential equations. Multidimensional problems monitored by partial differential equations, parametric problems with simple and multiple integrals, parametric problems of optimal control, and related questions of nonlinear integration will be presented elsewhere.

Chapter 1 is introductory. The many types of problems of optimization are reviewed and their intricate relationships illustrated.

Chapter 2 presents the necessary conditions, the sufficient conditions, and the value function theory for classical problems of the calculus of variations. In particular, the Weierstrass necessary condition is being studied as a necessary condition for lower semicontinuity on a given trajectory.

Chapter 3 consists mainly of examples. In particular, it includes points of Ramsey's theory of economic growth, and points of theoretical mechanics.

Chapters 4 and 5 deal with problems of optimal control. They contain a statement of the necessary condition, a detailed discussion of the transversality relation in its generality, a discussion of Bellman's value function theory, and a statement of Boltyanskii's sufficient condition in terms of regular synthesis.

Chapter 6 consists mainly of examples. In particular, points of the neoclassical theory of economic growth are also studied.

Chapter 7 presents two proofs of the necessary condition for problems of optimal control.

Chapter 8 contains preparatory material for existence theorems, in particular, Kuratowski's and Ryll-Nardzewski's selection theorems, McShane's and Warfield's implicit function theorem, and some simple forms of the lower closure theorem for uniform convergence.

Chapter 9 deals with existence theorems for problems of optimal control with continuous data and compact control space. These are essentially Filippov's existence theorems. The proofs in this chapter are designed to be elementary in the sense that mere uniform convergence is involved, whereas in Chapters 10 and 11 use is made of weak convergence in L_1 .

Chapter 10 presents the Banach-Saks-Mazur theorem, the Dunford-Pettis theorem, and closure, lower closure, and lower semicontinuity theorems for weak convergence in L_1 .

Chapter 11 deals with existence theorems based on weak convergence. Existence theorems are proved for Lagrange problems with an integrand which is an extended function, and then existence theorems are derived for problems of optimal control. Moreover, existence theorems are proved for problems with comparison functionals, for isoperimetric problems, and specifically for problems which are linear in the derivatives, or in the controls. In particular, this chapter contains a present day version of the theorem established by Tonelli in 1914 for problems with a uniform growth property.

In Chapter 12 existence theorems are presented where a growth assumption fails at the points of a "slender" set. In Chapter 13 existence theorems under numerous analytical conditions are studied. Chapter 14 deals with existence theorems for problems without growth assumptions. Chapter 15 presents theorems based on mere pointwise convergence. Chapter 16 deals with Neustadt-type existence theorems for problems with no convexity assumptions.

Chapter 17 covers a few points of convex analysis including duality, and the equivalence of a certain concept of upper semicontinuity for sets with

the concept of seminormality of Tonelli and McShane for functions, and suitable properties in terms of convex analysis.

Chapter 18 covers questions of approximation of usual and generalized trajectories.

Each chapter contains examples and exercises. Bibliographical notes at the end of each chapter provide some historical background and direct the reader to the literature in the field.

A number of parts in this book are in smaller print so as to facilitate, at a first reading, a faster perusal. The small-print passages include most of the examples and remarks, several of the complementary considerations, and a number of the more technical proofs.

I wish to thank the many associates and graduate students who, with their remarks and suggestions upon reading these notes, have contributed so much to make this presentation a reality.

Finally, I wish to express my appreciation to Springer-Verlag for their accomplished handling of the manuscript, their understanding and patience.

To Isotta, always

Contents

Chapter 1	
Problems of Optimization—A General View	1
1.1 Classical Lagrange Problems of the Calculus of Variations	1
1.2 Classical Lagrange Problems with Constraints on the Derivatives	3
1.3 Classical Bolza Problems of the Calculus of Variations	4
1.4 Classical Problems Depending on Derivatives of Higher Order	5
1.5 Examples of Classical Problems of the Calculus of Variations	5
1.6 Remarks	8
1.7 The Mayer Problems of Optimal Control	9
1.8 Lagrange and Bolza Problems of Optimal Control	11
1.9 Theoretical Equivalence of Mayer, Lagrange, and Bolza Problems of Optimal Control. Problems of the Calculus of Variations as Problems of Optimal Control	11
1.10 Examples of Problems of Optimal Control	14
1.11 Exercises	14
1.12 The Mayer Problems in Terms of Orientor Fields	15
1.13 The Lagrange Problems of Control as Problems of the Calculus of Variations with Constraints on the Derivatives	16
1.14 Generalized Solutions	18
Bibliographical Notes	23
Chapter 2	
The Classical Problems of the Calculus of Variations: Necessary Conditions and Sufficient Conditions; Convexity and Lower Semicontinuity	24
2.1 Minima and Maxima for Lagrange Problems of the Calculus of Variations	24
2.2 Statement of Necessary Conditions	30
2.3 Necessary Conditions in Terms of Gateau Derivatives	37
	ix

2.4	Proofs of the Necessary Conditions and of Their Invariant Character	42
2.5	Jacobi's Necessary Condition	53
2.6	Smoothness Properties of Optimal Solutions	57
2.7	Proof of the Euler and DuBois-Reymond Conditions in the Unbounded Case	61
2.8	Proof of the Transversality Relations	64
2.9	The String Property and a Form of Jacobi's Necessary Condition	65
2.10	An Elementary Proof of Weierstrass's Necessary Condition	69
2.11	Classical Fields and Weierstrass's Sufficient Conditions	70
2.12	More Sufficient Conditions	83
2.13	Value Function and Further Sufficient Conditions	89
2.14	Uniform Convergence and Other Modes of Convergence	98
2.15	Semicontinuity of Functionals	100
2.16	Remarks on Convex Sets and Convex Real Valued Functions	101
2.17	A Lemma Concerning Convex Integrands	102
2.18	Convexity and Lower Semicontinuity: A Necessary and Sufficient Condition	103
2.19	Convexity as a Necessary Condition for Lower Semicontinuity	104
2.20	Statement of an Existence Theorem for Lagrange Problems of the Calculus of Variations	111
	Bibliographical Notes	114

Chapter 3

Examples and Exercises on Classical Problems 116

3.1	An Introductory Example	116
3.2	Geodesics	117
3.3	Exercises	120
3.4	Fermat's Principle	120
3.5	The Ramsay Model of Economic Growth	123
3.6	Two Isoperimetric Problems	125
3.7	More Examples of Classical Problems	127
3.8	Miscellaneous Exercises	131
3.9	The Integral $I = \int (x'^2 - x^2) dt$	132
3.10	The Integral $I = \int x x'^2 dt$	135
3.11	The Integral $I = \int x'^2 (1 + x')^2 dt$	136
3.12	Brachistochrone, or Path of Quickest Descent	139
3.13	Surface of Revolution of Minimum Area	143
3.14	The Principles of Mechanics	149
	Bibliographical Notes	158

Chapter 4

Statement of the Necessary Condition for Mayer Problems of Optimal Control 159

4.1	Some General Assumptions	159
4.2	The Necessary Condition for Mayer Problems of Optimal Control	162
4.3	Statement of an Existence Theorem for Mayer's Problems of Optimal Control	173
4.4	Examples of Transversality Relations for Mayer Problems	174
4.5	The Value Function	181
4.6	Sufficient Conditions	184

4.7 Appendix: Derivation of Some of the Classical Necessary Conditions of Section 2.1 from the Necessary Condition for Mayer Problems of Optimal Control	189
4.8 Appendix: Derivation of the Classical Necessary Condition for Isoperimetric Problems from the Necessary Condition for Mayer Problems of Optimal Control	191
4.9 Appendix: Derivation of the Classical Necessary Condition for Lagrange Problems of the Calculus of Variations with Differential Equations as Constraints	193
Bibliographical Notes	195
 Chapter 5 Lagrange and Bolza Problems of Optimal Control and Other Problems	 196
5.1 The Necessary Condition for Bolza and Lagrange Problems of Optimal Control	196
5.2 Derivation of Properties (P1')—(P4') from (P1)—(P4)	199
5.3 Examples of Applications of the Necessary Conditions for Lagrange Problems of Optimal Control	201
5.4 The Value Function	202
5.5 Sufficient Conditions for the Bolza Problem	204
Bibliographical Notes	205
 Chapter 6 Examples and Exercises on Optimal Control	 206
6.1 Stabilization of a Material Point Moving on a Straight Line under a Limited External Force	206
6.2 Stabilization of a Material Point under an Elastic Force and a Limited External Force	209
6.3 Minimum Time Stabilization of a Reentry Vehicle	213
6.4 Soft Landing on the Moon	214
6.5 Three More Problems on the Stabilization of a Point Moving on a Straight Line	217
6.6 Exercises	218
6.7 Optimal Economic Growth	221
6.8 Two More Classical Problems	224
6.9 The Navigation Problem	227
Bibliographical Notes	232
 Chapter 7 Proofs of the Necessary Condition for Control Problems and Related Topics	 233
7.1 Description of the Problem of Optimization	233
7.2 Sketch of the Proofs	235
7.3 The First Proof	236
7.4 Second Proof of the Necessary Condition	256
7.5 Proof of Boltyanskii's Statements (4.6.iv–v)	264
Bibliographical Notes	269

Chapter 8	
The Implicit Function Theorem and the Elementary Closure Theorem	271
8.1 Remarks on Semicontinuous Functionals	271
8.2 The Implicit Function Theorem	275
8.3 Selection Theorems	280
8.4 Convexity, Carathéodory's Theorem, Extreme Points	286
8.5 Upper Semicontinuity Properties of Set Valued Functions	290
8.6 The Elementary Closure Theorem	298
8.7 Some Fatou-Like Lemmas	301
8.8 Lower Closure Theorems with Respect to Uniform Convergence	302
Bibliographical Notes	307
Chapter 9	
Existence Theorems: The Bounded, or Elementary, Case	309
9.1 Ascoli's Theorem	309
9.2 Filippov's Existence Theorem for Mayer Problems of Optimal Control	310
9.3 Filippov's Existence Theorem for Lagrange and Bolza Problems of Optimal Control	313
9.4 Elimination of the Hypothesis that A Is Compact in Filippov's Theorem for Mayer Problems	317
9.5 Elimination of the Hypothesis that A Is Compact in Filippov's Theorem for Lagrange and Bolza Problems	318
9.6 Examples	319
Bibliographical Notes	324
Chapter 10	
Closure and Lower Closure Theorems under Weak Convergence	325
10.1 The Banach–Saks–Mazur Theorem	325
10.2 Absolute Integrability and Related Concepts	326
10.3 An Equivalence Theorem	329
10.4 A Few Remarks on Growth Conditions	330
10.5 The Growth Property (ϕ) Implies Property (Q)	333
10.6 Closure Theorems for Orientor Fields Based on Weak Convergence	340
10.7 Lower Closure Theorems for Orientor Fields Based on Weak Convergence	342
10.8 Lower Semicontinuity in the Topology of Weak Convergence	350
10.9 Necessary and Sufficient Conditions for Lower Closure	359
Bibliographical Notes	364
Chapter 11	
Existence Theorems: Weak Convergence and Growth Conditions	367
11.1 Existence Theorems for Orientor Fields and Extended Problems	367
11.2 Elimination of the Hypothesis that A Is Bounded in Theorems (11.1. i–iv)	379
11.3 Examples	381
11.4 Existence Theorems for Problems of Optimal Control with Unbounded Strategies	383

11.5 Elimination of the Hypothesis that A Is Bounded in Theorems (11.4.i–v)	396
11.6 Examples	397
11.7 Counterexamples	398
Bibliographical Notes	399
 Chapter 12	
Existence Theorems: The Case of an Exceptional Set of No Growth	403
12.1 The Case of No Growth at the Points of a Slender Set. Lower Closure Theorems.	403
12.2 Existence Theorems for Extended Free Problems with an Exceptional Slender Set	411
12.3 Existence Theorems for Problems of Optimal Control with an Exceptional Slender Set	413
12.4 Examples	414
12.5 Counterexamples	415
Bibliographical Notes	415
 Chapter 13	
Existence Theorems: The Use of Lipschitz and Tempered Growth Conditions	417
13.1 An Existence Theorem under Condition (D)	417
13.2 Conditions of the F, G, and H Types Each Implying Property (D) and Weak Property (Q)	422
13.3 Examples	427
Bibliographical Notes	429
 Chapter 14	
Existence Theorems: Problems of Slow Growth	430
14.1 Parametric Curves and Integrals	430
14.2 Transformation of Nonparametric into Parametric Integrals	436
14.3 Existence Theorems for (Nonparametric) Problems of Slow Growth	438
14.4 Examples	440
Bibliographical Notes	442
 Chapter 15	
Existence Theorems: The Use of Mere Pointwise Convergence on the Trajectories	443
15.1 The Helly Theorem	443
15.2 Closure Theorems with Components Converging Only Pointwise	444
15.3 Existence Theorems for Extended Problems Based on Pointwise Convergence	446
15.4 Existence Theorems for Problems of Optimal Control Based on Pointwise Convergence	450
15.5 Exercises	451
Bibliographical Notes	452

Chapter 16	
Existence Theorems: Problems with No Convexity Assumptions	453
16.1 Lyapunov Type Theorems	453
16.2 The Neustadt Theorem for Mayer Problems with Bounded Controls	458
16.3 The Bang-Bang Theorem	460
16.4 The Neustadt Theorem for Lagrange and Bolza Problems with Bounded Controls	462
16.5 The Case of Unbounded Controls	464
16.6 Examples for the Unbounded Case	471
16.7 Problems of the Calculus of Variations without Convexity Assumptions	472
Bibliographical Notes	473
Chapter 17	
Duality and Upper Semicontinuity of Set Valued Functions	474
17.1 Convex Functions on a Set	474
17.2 The Function $T(x; z)$	478
17.3 Seminormality	481
17.4 Criteria for Property (Q)	482
17.5 A Characterization of Property (Q) for the Sets $\tilde{Q}(t, x)$ in Terms of Seminormality	486
17.6 Duality and Another Characterization of Property (Q) in Terms of Duality	488
17.7 Characterization of Optimal Solutions in Terms of Duality	496
17.8 Property (Q) as an Extension of Maximal Monotonicity	500
Bibliographical Notes	502
Chapter 18	
Approximation of Usual and of Generalized Solutions	503
18.1 The Gronwall Lemma	503
18.2 Approximation of AC Solutions by Means of C^1 Solutions	504
18.3 The Brouwer Fixed Point Theorem	508
18.4 Further Results Concerning the Approximation of AC Trajectories by Means of C^1 Trajectories	508
18.5 The Infimum for AC Solutions Can Be Lower than the One for C^1 Solutions	514
18.6 Approximation of Generalized Solutions by Means of Usual Solutions	517
18.7 The Infimum for Generalized Solutions Can Be Lower than the One for Usual Solutions	519
Bibliographical Notes	520
Bibliography	523
Author Index	537
Subject Index	540

CHAPTER 1

Problems of Optimization— A General View

1.1 Classical Lagrange Problems of the Calculus of Variations

Here we are concerned with minima and maxima of functionals of the form

$$(1.1.1) \quad I[x] = \int_{t_1}^{t_2} f_0(t, x(t), x'(t)) dt, \quad (') = d/dt,$$

where we think of $I[x]$ as dependent on an n -vector continuous function $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, or continuous curve of the form $C: x = x(t)$, $t_1 \leq t \leq t_2$, in R^{n+1} , in a suitable class. Actually the subject of our inquiry will go much farther than the mere analysis of minima and maxima of functionals.

Here t is the real or independent variable, $t \in R^1 = R$, usually called “time”, and $x = (x^1, \dots, x^n) \in R^n$, $n \geq 1$, is a real vector variable, usually called the *space* or *phase* variable. Thus, we deal with continuous functions $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, which we may call trajectories, or curves. Here $f_0(t, x, x')$ is a given real valued function defined on R^{1+2n} , or in whatever part of R^{1+2n} it is relevant and it will be called a *Lagrangian* function, or briefly a Lagrangian.

We may allow the variable (t, x) to vary only in a given set A of the tx -space R^{1+n} , possibly of the form $A = [t_0, T] \times A_0$, $A_0 \subset R^n$, and we do not exclude that A is the whole tx -space. Thus we may require that

$$(1.1.2) \quad (t, x(t)) \in A, \quad t_1 \leq t \leq t_2.$$

We may require the functions $x(t)$ to satisfy some boundary conditions. A typical one is “both end points fixed,” or $x(t_1) = x_1$, $x(t_2) = x_2$ (t_1, t_2, x_1, x_2 fixed), $t_1 < t_2$, $x_1 = (x_1^1, \dots, x_1^n) \in R^n$, $x_2 = (x_2^1, \dots, x_2^n) \in R^n$. We may

then say that we consider curves C “joining fixed points $1 = (t_1, x_1)$ and $2 = (t_2, x_2)$ in R^{1+n} ”.

A great variety of boundary conditions are of interest, e.g., C joins a fixed point $1 = (t_1, x_1)$ to a given curve $\Gamma: x = g(t)$, $t' \leq t \leq t''$, that is, $x(t_1) = x_1$, $x(t_2) = g(t_2)$, $t_1 < t_2$, $t' \leq t_2 \leq t''$. Alternatively, we may require that C join two given sets B_1 and B_2 in R^{n+1} . Thus, the boundary conditions concern the $2n + 2$ real numbers t_1 , $x(t_1) = (x_1^1, \dots, x_1^n)$, t_2 , $x(t_2) = (x_2^1, \dots, x_2^n)$, or the ends $e[x] = (t_1, x(t_1), t_2, x(t_2))$ of the trajectory x . Note that t_1 and t_2 , in particular, need not be fixed. Often, these boundary conditions are expressed in terms of a set of equalities or inequalities concerning the $2n + 2$ numbers above. A general and compact way to express boundary conditions is to define a subset B of the $t_1 x_1 t_2 x_2$ -space R^{2n+2} and to require that

$$(1.1.3) \quad e[x] \in B, \quad \text{or} \quad (t_1, x(t_1), t_2, x(t_2)) \in B.$$

Thus, the case of both end points fixed, or t_1, x_1, t_2, x_2 fixed, corresponds to B being the single point (t_1, x_1, t_2, x_2) in R^{2n+2} ; the case of fixed first end point (t_1, x_1) and second end point (t_2, x_2) on a given curve Γ corresponds to $B = (t_1, x_1) \times \Gamma$, a subset of R^{2n+2} .

Problems of minima and maxima for functionals (1.1.1) with only constraints as (1.1.2) and (1.1.3) are often referred to as Lagrange problems of the calculus of variations, and sometimes as free problems.

Besides (1.1.2), (1.1.3), another type of constraint is often required, namely

$$(1.1.4) \quad \int_{t_1}^{t_2} |x'(t)|^p dt \leq C$$

for some constants $p \geq 1$, $C > 0$. More generally, we may require that for some “comparison functional” we have

$$\int_{t_1}^{t_2} H(t, x(t), x'(t)) dt \leq C.$$

Alternatively, we may require that any number N of given analogous functionals have given values, say

$$J_j[x] = \int_{t_1}^{t_2} f_j(t, x(t), x'(t)) dt = C_j \text{ [or } \leq C_j], \quad j = 1, \dots, N.$$

These problems with equality signs are sometimes called *isoperimetric problems*. (See Section 3.6 for some examples). The same problems with \leq signs are sometimes called problems with comparison functionals.

And now a few words on the class of n -vector functions $x(t)$, $t_1 \leq t \leq t_2$, we shall take into consideration. One could expect to find the optimal solution in the class C^1 of all continuous functions $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, with continuous derivative $x'(t) = (x'^1, \dots, x'^n)$. Very simple examples (see e.g. Section 2.6, Remark 2) show that it would be more realistic to search for optimal solutions in the class, say C_s , of all continuous functions $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, with sectionally continuous derivative. In such a situation, if we assume that $f_0(t, x, u)$ is defined and continuous in

$A \times R^n$, then $f_0(t, x(t), x'(t))$ would be sectionally continuous in $[t_1, t_2]$ and (1.1.1) would be a Riemann integral.

However, in view of other examples (see e.g. Section 2.6, Remark 1) in which the optimal solution is not in such a class C_s , and particularly because of exigencies related to the existence theorems (Chapters 9–16), it has been found more suitable to search for optimal solutions in the larger class of all *absolutely continuous* (AC) n -vector functions $x(t) = (x^1, \dots, x^n)$. (See Section 2.1 for definitions, and the Bibliographical notes at the end of this Chapter for historical views).

We only mention here that the class of AC functions is the largest class of continuous functions $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, possessing derivative $x'(t) = (x'^1, \dots, x'^n)$ a.e. in $[t_1, t_2]$ and for which the fundamental theorem of calculus holds, i.e., $x(\beta) - x(\alpha) = \int_\alpha^\beta x'(t) dt$, the integral being a Lebesgue integral on each component (see Section 2.1 for the definition of AC functions). Conversely, if $g(t)$ is L -integrable, then $G(t) = \int_{t_1}^t g(\tau) d\tau$ is AC.

Again, if we assume that $f_0(t, x, u)$ is continuous in $A \times R^n$ and $x(t)$ is AC, then $f_0(\cdot, x(\cdot), x'(\cdot))$ is certainly measurable. In such a situation we shall explicitly require that $f_0(\cdot, x(\cdot), x'(\cdot))$ is L -integrable, and then (1.1.1) is an L -integral. We only mention here that a set E on the real line is said to be of measure zero if it can be covered by a countable collection of open intervals (α_i, β_i) , $i = 1, 2, \dots$, possibly overlapping, whose total length $\sum_i (\beta_i - \alpha_i)$ is as small as we want. A property P then is said to hold almost everywhere (a.e.) if it holds everywhere but at the points of a set E of measure zero.

1.2 Classical Lagrange Problems with Constraints on the Derivatives

A very important recent extension of the concept above is to consider the same integral (1.1.1), with the same possible constraint (1.1.2) and boundary conditions (1.1.3), but now with restrictions concerning the possible values of x' . This can be understood by saying that, for every $(t, x) \in A$, a subset $Q(t, x)$ of R^n is assigned, and that we consider only n -vector AC functions $x(t) = (x^1, \dots, x^n)$, $t_1 \leq t \leq t_2$, whose derivative $x'(t) = (x'^1, \dots, x'^n)$ must belong to the corresponding set $Q(t, x(t))$. In other words, we may require that the n -vector AC function $x(t)$ satisfy

$$(1.2.1) \quad x'(t) \in Q(t, x(t)), \quad t \in [t_1, t_2] \text{ (a.e.)}$$

This is called an *orientor field*, or an *orientor field relation*.

For instance, for $n = 1$ and $Q = Q(t, x) = [z | a \leq z \leq b]$, we would restrict ourselves to only those AC scalar functions $x(t)$ whose slope $x'(t)$ is between two fixed numbers a and b . For instance, for any $n \geq 1$ and $Q(t, x) = [z \in R^n | |z| \leq a]$, we would restrict ourselves to only those AC n -vector