Applications of Mathematics

17

Lamberto Cesari

Optimization—Theory and Applications

Problems with Ordinary Differential Equations

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With 82 Illustrations

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17

Edited by A. V. Balakrishnan

Preface

This book has grown out of lectures and courses in calculus of variations and optimization taught for many years at the University of Michigan to graduate students at various stages of their careers, and always to a mixed audience of students in mathematics and engineering. It attempts to present a balanced view of the subject, giving some emphasis to its connections with the classical theory and to a number of those problems of economics and engineering which have motivated so many of the present developments, as well as presenting aspects of the current theory, particularly value theory and existence theorems. However, the presentation of the theory is connected to and accompanied by many concrete problems of optimization, classical and modern, some more technical and some less so, some discussed in detail and some only sketched or proposed as exercises.

No single part of the subject (such as the existence theorems, or the more traditional approach based on necessary conditions and on sufficient conditions, or the more recent one based on value function theory) can give a sufficient representation of the whole subject. This holds particularly for the existence theorems, some of which have been conceived to apply to certain large classes of problems of optimization.

For all these reasons it is essential to present many examples (Chapters 3 and 6) before the existence theorems (Chapters 9 and 11–16), and to investigate these examples by means of the usual necessary conditions, sufficient conditions, and value function theory.

This book only considers nonparametric problems of the calculus of variations in one independent variable and problems of optimal control monitored by ordinary differential equations. Multidimensional problems monitored by partial differential equations, parametric problems with simple and multiple integrals, parametric problems of optimal control, and related questions of nonlinear integration will be presented elsewhere.

Chapter 1 is introductory. The many types of problems of optimization are reviewed and their intricate relationships illustrated.

Chapter 2 presents the necessary conditions, the sufficient conditions, and the value function theory for classical problems of the calculus of variations. In particular, the Weierstrass necessary condition is being studied as a necessary condition for lower semicontinuity on a given trajectory.

Chapter 3 consists mainly of examples. In particular, it includes points of Ramsey's theory of economic growth, and points of theoretical mechanics.

Chapters 4 and 5 deal with problems of optimal control. They contain a statement of the necessary condition, a detailed discussion of the transversality relation in its generality, a discussion of Bellman's value function theory, and a statement of Boltyanskii's sufficient condition in terms of regular synthesis.

Chapter 6 consists mainly of examples. In particular, points of the neoclassical theory of economic growth are also studied.

Chapter 7 presents two proofs of the necessary condition for problems of optimal control.

Chapter 8 contains preparatory material for existence theorems, in particular, Kuratowski's and Ryll-Nardzewski's selection theorems, McShane's and Warfield's implicit function theorem, and some simple forms of the lower closure theorem for uniform convergence.

Chapter 9 deals with existence theorems for problems of optimal control with continuous data and compact control space. These are essentially Filippov's existence theorems. The proofs in this chapter are designed to be elementary in the sense that mere uniform convergence is involved, whereas in Chapters 10 and 11 use is made of weak convergence in L_1.

Chapter 10 presents the Banach–Saks–Mazur theorem, the Dunford-Pettis theorem, and closure, lower closure, and lower semicontinuity theorems for weak convergence in L_1.

Chapter 11 deals with existence theorems based on weak convergence. Existence theorems are proved for Lagrange problems with an integrand which is an extended function, and then existence theorems are derived for problems of optimal control. Moreover, existence theorems are proved for problems with comparison functionals, for isoperimetric problems, and specifically for problems which are linear in the derivatives, or in the controls. In particular, this chapter contains a present day version of the theorem established by Tonelli in 1914 for problems with a uniform growth property.

In Chapter 12 existence theorems are presented where a growth assumption fails at the points of a "slender" set. In Chapter 13 existence theorems under numerous analytical conditions are studied. Chapter 14 deals with existence theorems for problems without growth assumptions. Chapter 15 presents theorems based on mere pointwise convergence. Chapter 16 deals with Neustadt-type existence theorems for problems with no convexity assumptions.

Chapter 17 covers a few points of convex analysis including duality, and the equivalence of a certain concept of upper semicontinuity for sets with the concept of seminormality of Tonelli and McShane for functions, and suitable properties in terms of convex analysis.

Chapter 18 covers questions of approximation of usual and generalized trajectories.

Each chapter contains examples and exercises. Bibliographical notes at the end of each chapter provide some historical background and direct the reader to the literature in the field.

A number of parts in this book are in smaller print so as to facilitate, at a first reading, a faster perusal. The small-print passages include most of the examples and remarks, several of the complementary considerations, and a number of the more technical proofs.

I wish to thank the many associates and graduate students who, with their remarks and suggestions upon reading these notes, have contributed so much to make this presentation a reality.

Finally, I wish to express my appreciation to Springer-Verlag for their accomplished handling of the manuscript, their understanding and patience.

To Isotta, always

Contents

CHAPTER 1
Problems of Optimization—A General View

1.1 Classical Lagrange Problems of the Calculus of Variations

Here we are concerned with minima and maxima of functionals of the form

$$I[x] = \int_{t_1}^{t_2} f_0(t, x(t), x'(t))\, dt, \qquad (') = d/dt, \tag{1.1.1}$$

where we think of $I[x]$ as dependent on an n-vector continuous function $x(t) = (x^1, \ldots, x^n)$, $t_1 \leq t \leq t_2$, or continuous curve of the form $C\colon x = x(t)$, $t_1 \leq t \leq t_2$, in R^{n+1}, in a suitable class. Actually the subject of our inquiry will go much farther than the mere analysis of minima and maxima of functionals.

Here t is the real or independent variable, $t \in R^1 = R$, usually called "time", and $x = (x^1, \ldots, x^n) \in R^n, n \geq 1$, is a real vector variable, usually called the *space* or *phase* variable. Thus, we deal with continuous functions $x(t) = (x^1, \ldots, x^n)$, $t_1 \leq t \leq t_2$, which we may call trajectories, or curves. Here $f_0(t, x, x')$ is a given real valued function defined on R^{1+2n}, or in whatever part of R^{1+2n} it is relevant and it will be called a *Lagrangian* function, or briefly a Lagrangian.

We may allow the variable (t, x) to vary only in a given set A of the tx-space R^{1+n}, possibly of the form $A = [t_0, T] \times A_0$, $A_0 \subset R^n$, and we do not exclude that A is the whole tx-space. Thus we may require that

$$(t, x(t)) \in A, \qquad t_1 \leq t \leq t_2. \tag{1.1.2}$$

We may require the functions $x(t)$ to satisfy some boundary conditions. A typical one is "both end points fixed," or $x(t_1) = x_1$, $x(t_2) = x_2$ $(t_1, t_2, x_1, x_2$ fixed), $t_1 < t_2$, $x_1 = (x_1^1, \ldots, x_1^n) \in R^n$, $x_2 = (x_2^1, \ldots, x_2^n) \in R^n$. We may then say that we consider curves C "joining fixed points $1 = (t_1, x_1)$ and $2 = (t_2, x_2)$ in R^{1+n}".

A great variety of boundary conditions are of interest, e.g., C joins a fixed point $1 = (t_1, x_1)$ to a given curve $\Gamma: x = g(t)$, $t' \le t \le t''$, that is, $x(t_1) = x_1$, $x(t_2) = g(t_2)$, $t_1 < t_2$, $t' \le t_2 \le t''$. Alternatively, we may require that C join two given sets B_1 and B_2 in R^{n+1}. Thus, the boundary conditions concern the $2n + 2$ real numbers t_1, $x(t_1) = (x_1^1, \ldots, x_1^n)$, t_2, $x(t_2) = (x_2^1, \ldots, x_2^n)$, or the *ends* $e[x] = (t_1, x(t_1), t_2, x(t_2))$ of the trajectory x. Note that t_1 and t_2, in particular, need not be fixed. Often, these boundary conditions are expressed in terms of a set of equalities or inequalities concerning the $2n + 2$ numbers above. A general and compact way to express boundary conditions is to define a subset B of the $t_1 x_1 t_2 x_2$-space R^{2n+2} and to require that

$$e[x] \in B, \quad \text{or} \quad (t_1, x(t_1), t_2, x(t_2)) \in B. \tag{1.1.3}$$

Thus, the case of both end points fixed, or t_1, x_1, t_2, x_2 fixed, corresponds to B being the single point (t_1, x_1, t_2, x_2) in R^{2n+2}; the case of fixed first end point (t_1, x_1) and second end point (t_2, x_2) on a given curve Γ corresponds to $B = (t_1, x_1) \times \Gamma$, a subset of R^{2n+2}.

Problems of minima and maxima for functionals (1.1.1) with only constrains as (1.1.2) and (1.1.3) are often referred to as Lagrange problems of the calculus of variations, and sometimes as free problems.

Besides (1.1.2), (1.1.3), another type of constraint is often required, namely

$$\int_{t_1}^{t_2} |x'(t)|^p \, dt \le C \tag{1.1.4}$$

for some constants $p \ge 1$, $C > 0$. More generally, we may require that for some "comparison functional" we have

$$\int_{t_1}^{t_2} H(t, x(t), x'(t)) \, dt \le C.$$

Alternatively, we may require that any number N of given analogous functionals have given values, say

$$J_j[x] = \int_{t_1}^{t_2} f_j(t, x(t), x'(t)) \, dt = C_j \; [\text{or } \le C_j], \qquad j = 1, \ldots, N.$$

These problems with equality signs are sometimes called *isoperimetric problems*. (See Section 3.6 for some examples). The same problems with $\le$ signs are sometimes called problems with comparison functionals.

And now a few words on the class of n-vector functions $x(t)$, $t_1 \le t \le t_2$, we shall take into consideration. One could expect to find the optimal solution in the class C^1 of all continuous functions $x(t) = (x^1, \ldots, x^n)$, $t_1 \le t \le t_2$, with continuous derivative $x'(t) = (x'^1, \ldots, x'^n)$. Very simple examples (see e.g. Section 2.6, Remark 2) show that it would be more realistic to search for optimal solutions in the class, say C_s, of all continuous functions $x(t) = (x^1, \ldots, x^n)$, $t_1 \le t \le t_2$, with sectionally continuous derivative. In such a situation, if we assume that $f_0(t, x, u)$ is defined and continuous in

$A \times R^n$, then $f_0(t, x(t), x'(t))$ would be sectionally continuous in $[t_1, t_2]$ and (1.1.1) would be a Riemann integral.

However, in view of other examples (see e.g. Section 2.6, Remark 1) in which the optimal solution is not in such a class C_s, and particularly because of exigencies related to the existence theorems (Chapters 9–16), it has been found more suitable to search for optimal solutions in the larger class of all *absolutely continuous* (AC) n-vector functions $x(t) = (x^1, \ldots, x^n)$. (See Section 2.1 for definitions, and the Bibliographical notes at the end of this Chapter for historical views).

We only mention here that the class of AC functions is the largest class of continuous functions $x(t) = (x^1, \ldots, x^n)$, $t_1 \le t \le t_2$, possessing derivative $x'(t) = (x'^1, \ldots, x'^n)$ a.e. in $[t_1, t_2]$ and for which the fundamental theorem of calculus holds, i.e., $x(\beta) - x(\alpha) = \int_\alpha^\beta x'(t)\,dt$, the integral being a Lebesgue integral on each component (see Section 2.1 for the definition of AC functions). Conversely, if $g(t)$ is L-integrable, then $G(t) = \int_{t_1}^t g(\tau)\,d\tau$ is AC.

Again, if we assume that $f_0(t, x, u)$ is continuous in $A \times R^n$ and $x(t)$ is AC, then $f_0(\cdot, x(\cdot), x'(\cdot))$ is certainly measurable. In such a situation we shall explicitly require that $f_0(\cdot, x(\cdot), x'(\cdot))$ is L-integrable, and then (1.1.1) is an L-integral. We only mention here that a set E on the real line is said to be of measure zero if it can be covered by a countable collection of open intervals (α_i, β_i), $i = 1, 2, \ldots$, possibly overlapping, whose total length $\sum_i (\beta_i - \alpha_i)$ is as small as we want. A property P then is said to hold almost everywhere (a.e.) if it holds everywhere but at the points of a set E of measure zero.

1.2 Classical Lagrange Problems with Constraints on the Derivatives

A very important recent extension of the concept above is to consider the same integral (1.1.1), with the same possible constraint (1.1.2) and boundary conditions (1.1.3), but now with restrictions concerning the possible values of x'. This can be understood by saying that, for every $(t, x) \in A$, a subset $Q(t, x)$ of R^n is assigned, and that we consider only n-vector AC functions $x(t) = (x^1, \ldots, x^n)$, $t_1 \le t \le t_2$, whose derivative $x'(t) = (x'^1, \ldots, x'^n)$ must belong to the corresponding set $Q(t, x(t))$. In other words, we may require that the n-vector AC function $x(t)$ satisfy

$$x'(t) \in Q(t, x(t)), \qquad t \in [t_1, t_2] \text{ (a.e.)} \tag{1.2.1}$$

This is called an orientor field, or an orientor field relation.

For instance, for $n = 1$ and $Q = Q(t, x) = [z \mid a \le z \le b]$, we would restrict ourselves to only those AC scalar functions $x(t)$ whose slope $x'(t)$ is between two fixed numbers a and b. For instance, for any $n \ge 1$ and $Q(t, x) = [z \in R^n \| z| \le a]$, we would restrict ourselves to only those AC n-vector