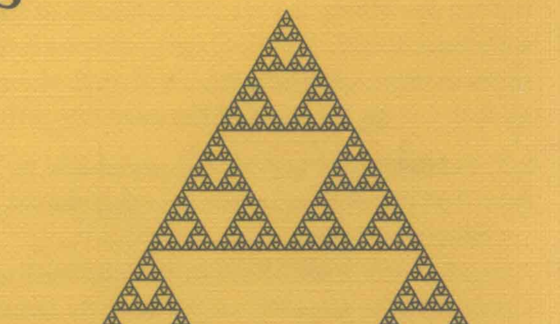
Lecture Notes in Mathematics

1567

R. L. Dobrushin S. Kusuoka

Statistical Mechanics and Fractals



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FOREWORD

The Nankai Institute of Mathematics held a special Year in Probability and Statistics during the academic year 1988-1989. We had over 150 specialists, professors and graduate students, who participated in this Special Year from August 1988 to May 1989. More than twenty outstanding probabilists and statisticians from several countries were invited to give lectures and talks. This volume contains two lectures, one is written by Professor R. L. Dobrushin, and the other one by Professor S. Kusuoka.

We would like to express our gratitude to Professors Dobrushin and Kusuoka for their enthusiasm and cooperation.

Ze-Pei Jiang
Shi-Jian Yan
Ping Cheng
Rong Wu

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ON THE WAY TO THE MATHEMATICAL FOUNDATIONS OF STATISTICAL MECHANICS

R.L.DOBRUSHIN

§0. Introduction

When I was a student in Moscow University at the end of the forties, I had to attend some lectures on physics. I had at that time a deep impression that although the content was very interesting to me, the form seems rather formidable. I asked myself, "Why don't they distinguish definitions from implications?" "Do they really fail to understand the difference between the necessary and the sufficient conditions?" "How can they formulate statements for which we see evident counterexamples?" I hoped that if I should become a professor, it would be possible for me to give a course of lectures on physics at a logical level consistent with the standard set by modern mathematics.

Later I understood that I had been naive and that the situation is not so simple. In fact my professors were not very bad. The style of their lectures reflected the logical level of modern theoretical physics which contrasts sharply with the logical level of modern mathematics. However, this was not always so. In the last century mathematics and physics were almost united. Readers will easily recall the names of great scientists who made important contributions both to physics and to mathematics. At that time there were no essential differences between the styles of exposition in the two subjects. Mathematicians and physicists spoke the same language and understood each other.

At the beginning of this century physics and mathematics began to move in different directions. Mathematics was incorporating very exciting new ideas: set theory, measure theory, modern algebra, functional analysis, topology, etc. New and higher standards of mathematical rigor were developed and any purported mathematical result which did not conform to this standard was considered either as erroneous or at least as lying outside of mathematics. We have now a standard universally accepted language for modern mathematics.

Physics went along another path. The new exciting ideas of quantum physics, theory of relativity, statistical physics, etc. posed attractive problems which required urgent solutions. In the beginning the methods of classical mathematical analysis developed in the last century were enough for their purposes. Physicists did not know modern mathematics and often treated it as an abstract and useless game. I have heard, for example, that our great physicist, Landau, said that he could invent all the mathematics which he needed. Such a point of view was even fashionable among physicists during that period. Physicists did not want to waste time on the fussy mathematical details needed for a rigorous proof. They considered something to be "proved" by using an argument which to a mathematician was just a rough plan or an idea for a future proof.

As a result, mathematicians and physicists almost ceased to understand and even hear each other.

A particular example of this estrangement is provided by probability theory (including the theory of random processes) on the one hand and statistical mechanics on the other. Both subjects were developing actively during these years, but even though they are concerned essentially with the same questions they were isolated from each other. And, for the most part, practitioners of the two subjects almost forget each other's existence.

In the middle of this century the situation began to change with the impulse for change coming from both mathematics and physics. Many good mathematicians began to realize that although there were still open and difficult problems in traditional mathematical directions the main constructions had been completed, and it was time for mathematics to have an infusion of new fresh ideas and problems. They turned to physics for inspiration. On the other hand, the constructions of modern physics became more and more complex and abstract. Unexpectedly, modern mathematics found applications in modern physics. The ideas of modern algebra and differential topology became essential to relativity theory and the theory of quantum fields. The ideas of functional analysis are basic for quantum mechanics, etc.

Interacton between the theory of probability and statistical physics also began to develop very rapidly. I think the investigations of recent years reveal that from the mathematical point of view statistical physics can be considered as a branch of probability theory. Seemingly all the main ideas and problems of statistical physics can be formulated in probabilitic language. But, of course, only a small portion of the assertions in statistical physics can be proved now at a mathematically rigorous level.

One should not suppose that all physicists will adopt the standards of mathematical rigor in pursuing their studies. In their obvious anxiety for quick results they will not cease to neglect mathematical logic. However, beginning in the fifties a different discipline with great rigor was evolved; it is the new science of mathematical physics. This is not the earlier "mathematical physics" which, for the most part, constituted a chapter in partial differential equations; but it is a science which is distinguished from physics and mathematics and lies between them. This new mathematical physics uses the language and standards of modern mathematics in studying the problems of physics. There are many scientists trained both in physics and mathematics who now work in this area. They also take up the most important task of helping mathematicians and physicists understand the problems and results in their respective fields in terms of what is apprehensible to both of them. Aside from several journals specializing in mathematical physics, there is now an international organization separate from the traditional physics and mathematics organizations.

The mathematical statistical physics about which I will speak in my lectures here has to be considered as a branch of mathematical physics strongly connected with probability theory, and I will speak only about the classical statistical physics. Classical theory means that it does not use the notions of quantum mechanics. However, all the ideas of classical statistical physics have their analogue in quantum statistical physics. Sometimes one refers to the area of mathematics used to study quantum mechanics problems as the non-commutative probability theory.

Statistical physics is strongly connected with other important branches of physics.

Thus, quantum field theory, which unites quantum mechanics and relativistic theory, can also be transformed into probabilistic language. To do this it is necessary to analytically extend quantum field theory to the case of complex time parameter, and to consider the case of pure imaginary time. Then we obtain a probabilistic picture of the so-called Euclidean field theory. Its connection with statistical physics is the same as that of continuous time random processes with discrete time random processes. Statistical physics can be used as a discrete approximation to quantum field theory, but the continuous version is much more complicated. In Euclidean quantum field theory it is necessary to consider Markov random fields in which realizations are distributions (generalized functions). This is not surprising from the point of view of classical probability theory. Every probabilist knows that almost all trajectories of a Markov diffusion process are continuous but non-differentiable functions. In the multidimensional case studied in quantum field theory the realizations of the natural Markov field become even worse. This very interesting theme requires a special exposition which we will not give here.

The aim of these lectures is to give an exposition, at a mathematical level, of the foundations of classical statistical mechanics. It is not easy even now. As a result of mathematical investigations in recent years we can at least reformulate all of the main notions of statistical mechanics in the language of mathematical definitions. But from the point of view of mathematicians modern statistical physics is something like a mix of some continents of well-developed mathematical theories with islands of separate mathematical results, amid a sea of open problems and conjectures (Of course, most physicists think of conjectures as results). Each year more and more conjectures get transformed into theorems. But now the majority of mathematical papers are devoted to problems of equilibrium statistical physics. Progress in this domain has found a systematic exposition in book form (See, for example, [Sinai(1982)], [Georgii(1988)]). The problem of the foundation of statistical physics, including the foundation of equilibrium statistical physics, is in the realm of nonequilibrium statistical physics, and here we have only isolated islands of theorems in a sea of conjectures.

Nevertheless, it seems that we now see a plan, a way to construct an orderly theory. I will try to give the main mathematical definitions, and explain the physical ideas underlying these definitions. I will also formulate a lot of open mathematical problems. Many of them seem very difficult now. I will also formulate theorems whenever they exist, but I will rarely give nontrivial proofs, leaving proofs to be found in the references. As is usual with young branches of mathematics all the proofs in mathematical nonequilibrium statistical physics are very complex and involved. Usually, with the development of a branch of mathematics the proofs become simpler and shorter. Since this is not so yet in the area discussed here it is not possible to give systematic proofs on the scale of these lectures.

I hope that the publication of these lectures helps to stimulate mathematical investigations in this field, especialy in China where I see a lot of talented young mathematicians who are eager to work on new problems. I am very grateful to Prof. Chen Mu-fa and his colleagues Chen Dong-ching and Zheng Jun-li who wrote up my lectures and helped to prepare their final version. Without their invaluable and well-qualified help this text would have never been written. I am also grateful to the members of the Nankai Institute of Mathematics for their hospitality. Here in Tianjin I have a happy

possibility to meet Prof. M.D.Donsker.* I am very grateful to him for translating this introduction from its original Russian-Chinese dialect into real English.

Tianjin, 1987 December.

*which passed away prematurely in 1991.

§1. Realizations of the Classical Fluid Model

In this lecture I will speak mainly about the classical fluid model where the dynamics of particles is governed by the laws of classical Newtonian dynamics. It is the most natural and best-known model of statistical physics. Of course, as many physical models are, it is only an approximation to reality. For example, it does not take into account the quantum effects.

We will assume for simplicity that all particles are similar, i.e. they are particles of the same substance. The generalization to the case of particles of several types is not so complex. Denote by $(q, v) \in \mathbb{R}^d \times \mathbb{R}^d$ the particle with position q and velocity v, where d is a positive integer. In classical physics, $d = 3$. But some other dimensions also have physical interest (Dimension $d = 1$ corresponds to the statistics of threads, dimension $d = 2$ corresponds to the statistics of surfaces, dimension $d = 4$ corresponds to the problems arising from quantum field). So we will suppose that d is arbitrary and will, as in modern physics, follow the change of situation in the dependence of d.

If we have N particles, we denote their configuration by

$$\omega = ((q_1, \nu_1), \cdots, (q_N, \nu_N)) \in \hat{\Omega}_N \triangleq (\mathbb{R}^d \times \mathbb{R}^d)^N.$$

The realization space with an arbitrary number of particles is defined as follows:

$$\hat{\Omega} = \cup_{N=0}^{\infty} \hat{\Omega}_N$$

where $\hat{\Omega}_0$ is an empty set of particles.

Finally, from the physical point of view, it is natural to treat particles as undistinguished ones. For any $\omega \in \hat{\Omega}, \omega = ((q_1, \nu_1), \cdots, (q_N, \nu_N))$ and $A \subset \mathbb{R}^d \times \mathbb{R}^d$, let

$$\pi_\omega(A) = |\{i \in \{1, \cdots, N\} : (q_i, \nu_i) \in A\}|$$

where $|A|$ = the number of elements in A, π_ω is an integer-valued measure on Borel σ-algebra in $\mathbb{R}^d \times \mathbb{R}^d$ and $\pi_\omega(\mathbb{R}^d \times \mathbb{R}^d) < \infty$. In this way, we have defined a mapping from $\hat{\Omega}$ into the space $\hat{\Pi}$ of such measures, we call $\pi \in \hat{\Pi}$ an ordinary realization if $\pi(x) = \pi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d \times \mathbb{R}^d$. Let

$$X_\pi = \{x \in \mathbb{R}^d \times \mathbb{R}^d : \pi(x) = 1\}.$$

So we can interpret the ordinary realization X_π as a finite subset of the space $\mathbb{R}^d \times \mathbb{R}^d$. For most of the situations it is enough to consider only ordinary realizations and we will do so in almost all our lectures.

Statistical physics studies a finite but very large system of particles. One of the main features of mathematical approach is the explicit consideration of an infinite particle system which makes many notions of statistical physics much more sharp and accurate. We will systematically use in these lectures such a point of view. Let

$\Pi \triangleq \{\pi : \pi$ is an integer-valued measure on $\mathbb{R}^d \times \mathbb{R}^d$ with $\pi(S \times \mathbb{R}^d) < \infty$ for all compact subsets $S \subset \mathbb{R}^d\}$

$\Omega \triangleq \{X \subset \mathbb{R}^d \times \mathbb{R}^d : |X \cap (S \times \mathbb{R}^d)| < \infty$ for all compact subsets $S \subset \mathbb{R}^d\}$, We will call elements of Π and Ω locally finite realizations.

We assert that $\Omega \subset \Pi$ by using the identification similarly used above for a finite particle system.

Let $\mathcal{B}_\Pi$ be the smallest σ-algebra with respect to which the functions $f(\pi) = \pi(S \times \tilde{S})$, $\pi \in \Pi$ are measurable, where S and $\tilde{S}$ are compact subsets of $\mathbb{R}^d$. Let $\mathcal{B}_\Omega$ be the smallest σ-algebra with respect to which the functions $f(X) = X \cap (S \times \tilde{S})$, $X \subset \Omega$ are measurable for all pairs $(S, \tilde{S})$ of compact subsets of $\mathbb{R}^d$. It is easy to prove the following fact:

1) $\Omega \subset \Pi$ is a measurable subset of Π, and the restriction of $\mathcal{B}_\Pi$ on Ω coincides with $\mathcal{B}_\Omega$.

We will leave the proof of this fact to our reader as an exercise.

Given a compact subset $V \subset \mathbb{R}^d$, we define Π_V and Ω_V by replacing $\mathbb{R}^d \times \mathbb{R}^d$ with $V \times \mathbb{R}^d$ in the definitions of Π and Ω respectively. Similarly we introduce $\mathcal{B}_{\Pi_V}$ and $\mathcal{B}_{\Omega_V}$. Of course $\Omega_V \subset \Omega$, $\Pi_V \subset \Pi$ and it is easy to check that these embeddings are measurable.

Now we have two measurable spaces $(\Omega, \mathcal{B}_\Omega)$ and $(\Pi, \mathcal{B}_\Pi)$. We will construct a basic measure on them, connected with the usual Lebesgue measures in Euclidean space. Define $\hat{\Omega}_V = \cup_{N=0}^{\infty} \hat{\Omega}_V^N$ where $\hat{\Omega}_V^N = (V \times \mathbb{R}^d)^N$, and the transformation $\alpha : \hat{\Omega}_V \to \Pi_V$, $\alpha(\omega) = \pi_\omega(\cdot) \in \Pi_V, \omega \in \hat{\Omega}_V$.

Let $\hat{\lambda}_N$ be the Lebesgue measure on $(V \times \mathbb{R}^d)^N, N = 1, 2, \cdots$. Define a measure $\hat{\lambda}$ on $\hat{\Omega}_V$ such that $\hat{\lambda}(A) \triangleq \hat{\lambda}_N(A)$ for $N \geq 1$ and all measurable subsets $A \subset (V \times \mathbb{R}^d)^N$, and set

$$\Pi_V^N = \{\pi \in \Pi_V : \pi(V \times \mathbb{R}^d) = N\}, \Pi_V = \cup_{N=0}^{\infty} \Pi_V^N.$$

For $A \subset \Pi_V^N, N > 0$, we define

$$\lambda(A) \triangleq \frac{1}{N!} \hat{\lambda}(\alpha^{-1}(A))$$

and assume $\lambda(\Pi_V^0) = 1$ (The set Π_V^0 consists of a unique measure $\pi_V^0 \equiv 0$). For any $B \subset \Pi_V$, there is a partition of B, $B = \sum_{i=0}^{\infty} A^i, A^i \in \Pi_V^i, i = 0, 1, 2, \cdots$, so we can define

$$\lambda(B) \triangleq \sum_{i=0}^{\infty} \lambda(A^i).$$

For any compact subsets $V_1, V_2, V_1 \subset V_2 \subset \mathbb{R}^d \times \mathbb{R}^d$, the restriction of λ_{V_2} to V_1 is equal to λ_{V_1}. This is because of the consistency of Lebesgue measures. Since $\Pi = \cup_{V \subset \mathbb{R}^d} \Pi_V$, then using the previous property we can define a measure on Π which is also denoted by λ and will be called the basic measure on Π. By definition, we have

2). If $V_1, V_2 \subset \mathbb{R}^d$ are compact subsets and $V_1 \cap V_2 = \phi, V \triangleq V_1 \cup V_2$, then $\Pi_V = \Pi_{V_1} \times \Pi_{V_2}$ $\mathcal{B}_{\Pi_V} = \mathcal{B}_{\Pi_{V_1}} \times \mathcal{B}_{\Pi_{V_2}}$ and $\lambda_V = \lambda_{V_1} \times \lambda_{V_2}$.

3). $\lambda(\Pi \setminus \Omega) = 0$.

The proof of these facts can also be considered as an exercise. Very often we will treat the basic measure as a measure on the space of ordinary realizations Ω. The reader who knows well the probability theory understand that λ is a Poisson measure well known in the theory of point random fields.

§2. Dynamics of a Finite System

Suppose that we are given N particles $\omega = ((q_1, \nu_1), \cdots, (q_N, \nu_N))$ and an interacting potential U. Here we consider only the pair potentials which are translation invariant, isotropic, and so we interpret a potential as a function on $\mathbb{R}^+ = \{x \in \mathbb{R} : 0 < x < \infty\}$ into $\mathbb{R}$. We will consider the following equations of motion of Newtonian type.*

$$(2.1) \qquad \begin{cases} \frac{dq_i(t)}{dt} = \nu_i(t) & i = 1, \cdots, N, t \in (0, \infty) \\ \frac{d\nu_i(t)}{dt} = -m \; grad_{q_i} \sum_{j=1, j\neq i}^{N} U(|q_j - q_i|), \end{cases}$$

where m is the mass of one particle. If we denote the momentum by $p_i = m\nu_i$, we have the following Hamiltonian equations:

$$\begin{cases} \frac{dq_i}{dt} = grad_{p_i} H(q_1, \cdots, q_N, p_1, \cdots, p_N) \\ \frac{dp_i}{dt} = -grad_{q_i} H(q_1, \cdots, q_N, p_1, \cdots, p_N) \end{cases} \qquad i = 1, \cdots, N.$$

where

$$H(q_1, \cdots, q_N, p_1, \cdots, p_N) \\ = \frac{1}{2} \sum_{i=1}^{N} \frac{(p_i)^2}{m} + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} U(|q_i - q_j|).$$

This last quantity is called the Hamiltonian of the system. In the following we let $m = 1$.

Mathematicians often ask: What function U is a real physical potential? The question is not correct. First of all, any classical model is only a rough approximation to a quantum model, and our choice of potential is such an approximation in some sense. Secondly, there are a lot of types of particles, and different types of potentials are naturally for different types of particles. Finally, it is better to have results for some potential than to have no results. So potentials having the simplest analytical structures are often considered. But the results for any potentials are interesting. It is especially interesting to have results applicable to a wider class of potentials and to follow the change of qualitative property of the system in the dependence on the potential. This conclusion should sound pleasant to mathematicians.

Now we will give some typical examples of the potentials.

1. Lenard-Jons potential

$$U(x) = \begin{cases} \frac{K_1}{|x|^n} - \frac{K_2}{|x|^l}, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0, \end{cases}$$

where K_1 and K_2 are positive constant, l and n are positive integers. In the 3-dimensional case, it is often to suppose that

$$l = 6, \qquad n = 2l = 12.$$

Figure 1 indicates one of this kind of potentials.

* See for example [Arnold(1978)] in connection with elementary notions of mechanics used in these lectures.

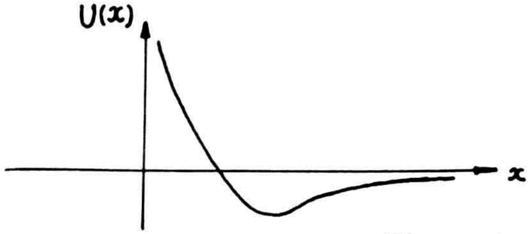


Figure 1

This structure of the potential can be justified by some quantum type of consideration for the case of one-atom gas. The decreasing part of the graph corresponds to the repulsion of particles and the increasing part corresponds to their attraction. The value $U(0) = \infty$ means that two particles can not collide with each other.

2. Morse potential

$$U(x) = K[1 - \exp\{-\alpha(x - \hat{x})^2\}]^2, \quad x \geq 0,$$

where α and K are constants; $\hat{x}$ is a fixed reference point.

Figure 2

Such a potential is used for two-atom particles. The value $U(0) < \infty$ means that the two particles can meet together. Of course this is not very natural from the physical point of view.

3. Hard core potential

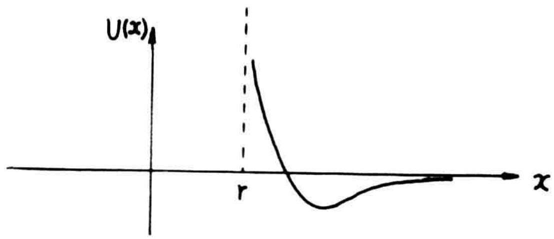
Suppose that there exists an $r > 0$ such that

$$U(x) = \infty, \qquad |x| \leq r;$$

or, in another language, each particle is a hard sphere of diameter r. It means that particles can not be closer to one another than at a distance r.

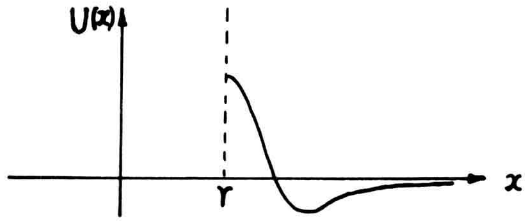
Here we give three potentials of this kind.

3(a). $U(x) \to \infty$ as $x \to r$:

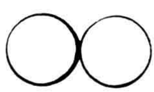


In this case particles can not collide because of repulsion. When the distance between them is close to r there is a very strong repulsion.

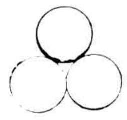
3(b). $U(x) \to c \neq \infty$ as $x \to r$:



In this case particles can collide. So we have to add to the equations of motion (2.1) some boundary conditions.
Usually we will suppose that the collision is elastic one (As for ordinary billiard-balls, see later). To have only pair collisions will permit us to define a unique solution to the equations of motion. This is not so in the case of multiple collisions (See graph below).



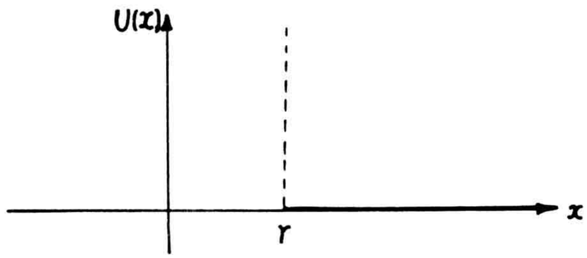
Pair collision



Collision of three particles.

It is natural to suppose that under ordinary circumstances the multiple collisions occur with probability 0, but this question has not been thoroughly studied at a mathematical level.

3(c). $U(x) \equiv 0$, $|x| > r$:



It is the case of pure hard core potential where interaction of particles arises only at the moment of their collisions.

Generally we will suppose that $r = 0$ is a possible value for the hard core diameter. The case

$$U(x) \equiv 0, x \in \mathbb{R}^+$$

corresponds to that of an ideal gas where particles do not interact.

We shall mainly suppose that some conditions of smoothness are true:

1^0 The smooth potential. The potential $U(x)$ has continuous first derivative in $x \in (0, \infty)$. In this case we can find a unique solution to the equations of motion under certain initial conditions.

2^0 The smooth hard core potential, i.e.

$$U(x) = \infty, \quad |x| \leq r,$$
$$U(x) \to \infty, \quad \text{as } x \to r,$$

and $U(x)$ has continuous first derivative for $x \in (r, \infty)$. This means that when particles get closer and closer, the energy becomes very great. By using the law of conservation of energy we can again prove the existence of a unique solution.

Now we give some notations. In the following we consider only ordinary realization, and in an obvious way redefine the motion as a motion of nondistinquished particles.

Let $\mathcal{A}$ denote the set of all finite subsets of $\mathbb{R}^d \times \mathbb{R}^d$. For each $t \in \mathbb{R}_+ = [0, \infty)$, we define a mapping $T_t : \mathcal{A} \to \mathcal{A}$ as follows: $T_t a$ is the realization at the moment t if $a \in \mathcal{A}$ is an initial realization, i.e. $T_t a = a(t)$, where $a(t)$ is the solution of the equations of motion with initial condition a. Using the well-known properties of the solution of differential equations, we can define also T_t for $t < 0$ and have $T_{t+s} = T_t \cdot T_s$ for all $s, t \in \mathbb{R}$. So $\{T_t : t \in \mathbb{R}\}$ is a group of transformations if, of course, restrictions discussed above are valid.

In order to describe the laws of conservation, we have to introduce additional functions. Let ϕ be a function on $\mathbb{R}^d \times \mathbb{R}^d$. For any $a \in \mathcal{A}$, we define $F : \mathcal{A} \to \mathbb{R}^d, (d = 1, 2, \cdots)$ by setting

$$F(a) = \sum_{(q,\nu)\in a} \phi(q,\nu).$$

We call this kind of F the (translation-invariant) first-order additive functional on the realization space.

We give some examples of the additive functional.
Example 1.

$$N(a) = \sum_{(q,\nu)\in a} 1, \quad \phi(\cdot) \equiv 1.$$

This is the total number of particles.
Example 2.

$$M(a) = \sum_{(q,\nu)\in a} \nu, \quad \phi(q,\nu) = \nu.$$

This is the momentum of the system.

When $F(a(t)) \equiv const.$ for any initial realization $a(0)$, we have a law of conservation. So, in Example 1, it is the law of conservation of particles (or of the mass); in Example 2, it is the law of conservation of momentum. These laws of conservation for the dynamical system of finite particles are well-known from elementary courses in mechanics.

If ϕ is defined on $(\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d)$, then

$$F_2(a) = \sum_{(q,\nu_1),(q_2,\nu_2)\in a} \phi(q_1,\nu_1; q_2,\nu_2),$$

is called an additive functional of second order. We will call this functional translation-invariant if

$$\phi(q_1,\nu_1; q_2,\nu_2) = \phi(q_1 - q_2, \nu_1, \nu_2).$$

We can also define a translation-invariant additive functional of any order like this.

The well-known law of conservation of energy can be described by the following translation-invariant additive functional of second order.

$$E(a) = \frac{1}{2} \sum_{(q,\nu)\in a} \nu^2 + \sum_{(q_1,\nu_1),(q_2,\nu_2)\in a: q_1 \neq q_2} U(|q_1 - q_2|)$$

where U is the potential defined previously. For the case when collisions are possible it is well known from mechanics that the laws of conservation of momentum and of energy are true when the colliding of particles can be considered as elastic collisions.

There exist non-translation-invariant laws of conservation. For example, the law of conservation of central momentum

$$F(a) = \sum_{(q,\nu)\in a} [q, \nu],$$

where $[\cdot, \cdot]$ is the scalar product in Euclidean space. Such non-translation-invariant conservation laws are not essential for the problems concerning the foundation of statistical mechanics.

There are some degenerate systems for which we have a lot of additional translation-invariant laws of conservation. For example, this is the case if dimension $d = 1$ and

$$U(x) = c(shAx)^{-2};$$

and also the limiting case $A \to 0, c/A^2 \to const$ with the potential

$$U(x) = cx^{-2}.$$

Here we have an infinite system of non-trivial translation- invariant laws of conservation.

Another example gives the pure hard core potential for dimension $d = 1$. We call the corresponding system as one of 1-dimensional hard rods. Here at the moment of collision the two particles simply exchange their velocities. So for any function $\phi(\nu)$ the relation

$$F_\phi(a) = \sum_{(q,\nu)\in a} \phi(\nu)$$

gives a translation-invariant law of conservation.

An important hypothesis states that the desribed cases are only exceptional cases, and (may be under some mild additional hypothesis of a general type) for all other potentials there are no additional laws of conservation. Under some strong additional conditions about potentials and functionals this important hypothesis has been proved by [Gurevich, Suhov 1976, 1982].

The structure of additive functionals plays a very important role in the description of the structure of equilibrium states. In both degenerated models described above the last structure also has a special form, see §4 for hard rod system and see [Chulaevsky, 1983] for the system with potential cx^{-2}.

We need to introduce other important properties of finite particle systems. For any $A \in \mathcal{A}$, it follows from Liouville theorem that

$$\lambda(A) = \lambda(T_t A), \quad t \geq 0$$

where measure $\lambda(\cdot)$ was defined in §1. So we have a dynamic system with an invariant measure. The other property is time-reversibility. This means that

$$T_{-t}a = (T_t a^*)^*, \quad t \in \mathbb{R}^1, \quad a \in \mathcal{A},$$

where $a = \{(q, \nu)\} \to a^* = \{(q, -\nu)\}$.