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S L Campbell (Editor)

# Recent applications of generalized inverses

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# 1 Introduction

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# Recent applications of generalized inverses

## 1. INTRODUCTION

This volume had its genesis at a special section during the 1976 AMS Regional Conference held at Columbia, South Carolina. This section was set up by Muir Z. Nashed of the University of Delaware, Newark, Delaware, and chaired by Carl D. Meyer, Jr., of North Carolina State University, Raleigh, North Carolina.

Originally, there was to be a published conference proceedings. Between the time of the Columbia meeting and its appearance in 1982, the volume's rationale, purpose, content, and editorship have changed. The remainder of this section will discuss the scope and intent of this volume in its current form. The remainder of this paper will discuss how the papers in this volume fit in with the current state of the theory of generalized inverses. Some reference will be made to recent work not covered by this volume. The referencing is not intended to be complete. The reader interested in further study of any of these topics is referred to the bibliographies of the cited papers. Cited papers which appear in this volume are denoted by double brackets[[ ]].

There exist several volumes on generalized inverses prior to 1976, (see for example) [1], [6], [10], [12], [13], [54], [60]. In 1976 there appeared an excellent and extensive survey volume [53] with an almost exhaustive bibliography.

The mid-1970's have seen somewhat of a change in the direction and type of research done on and with generalized inverses. Prior to this period, research was often concerned with equation solving ( (1) - inverses) and least squares inverses. This reflects, in part, the original impetous for the study of generalized inverses in statistics. The relationship between generalized inverses and statistics is still of some interest [56]. However, increasing amounts of research have been done on such topics as; infinite dimensional theory, numerical considerations, matrices of special type (boolean, integral), matrices over algebraic structures other than the real or complex numbers, systems theory, and non-equation solving inverses. Of course, all of these topics have their roots in the 1960's and some, such as the first two, are well represented in [12], [53]. The point is that the current flavor of research on generalized inverses has changed.

It is the intent of this volume to capture this "flavor" by publishing original, "state of the art" papers that will not appear elsewhere, the majority of which relate to "recent applications". Some of the original papers presented at the 1976 conference were judged to be still current and have been included in their original form. Any that may have become dated since that conference have been revised as of the Fall of 1981. In addition, new papers were solicited during 1981 to fill gaps in this volume's coverage.

Traditional applications, such as least squares analysis in finite dimensional spaces are well covered in earlier volumes, and will not be repeated here though interest continues [38], [48]. This volume is designed to complement and update earlier works and not to supplement or replicate them. The coverage is, of course, not exhaustive and is reflective of both interests of the editor and the requirement that the work not appear elsewhere.

Finally, it should be pointed out that generalized inverses, like the ordinary inverse, are frequently not essential to the development. However, they often provide a clarifying conceptual tool that adds insight and simplifies the development. There are two exceptions the author is aware of where generalized inverses provide simple expressions for projections for which alternative simple algebraic formula do not seem to exist. One is the characterization of the orthogonal projection onto the intersection of two subspaces due to Anderson and discussed in [5]. The second is the characterization of the subspace of consistent initial conditions for a singular linear system of differential equations as $\hat{A}^D\hat{A}$ which is due to Campbell, Meyer, and Rose [14]. Related to this last application is the work of Meyer on Markov Chains [14], [48], [[50]], [51].

In many problems involving singular behavior of some kind, the ideas of a generalized inverse are implicit. This volume is concerned only in those developments where the generalized inverse appears explicitly and plays a major role.

## 2. DIFFERENTIAL AND DIFFERENCE EQUATIONS

One area of current research on the applications of generalized inverses deals with singular systems of (nonlinear) differential equations of the form

$$A\dot{\underline{x}} + \underline{B}(\underline{x}) = \underline{f}. \quad (1)$$

Such equations arise in singular perturbations, cheap control problems, and descriptor systems. The basic material on the application of the Drazin inverse to (1) may be found in [14], [16]. In this work the Drazin inverse makes possible a simple characterization of certain subspaces which would ordinarily be defined by a iterative procedure.

Additional results and a fairly complete bibliography may be found in [17] which will appear in 1982. Accordingly this topic will not be explicitly covered in this volume. However, the papers of Campbell [[18]] and Bouldin [[11]] do discuss some of the problems in extending the results of [14], [16] to infinite dimensional spaces. Wilkinson's paper [[69]] explains how one might actually implement these ideas in solving linear systems in the form (1). He also discusses how one can compute the Drazin inverse.

The equation solving inverses ( (1) - inverses and Moore-Penrose inverses) are sometimes used in the context of differential equations, usually in the systems and control literature. See [42], [45] for example, and [16], [17], [19], [25], [46], [47].

Differential equations can also be viewed as operator equations in infinite dimensional spaces. This will be commented on shortly.

## 3. ITERATIVE PROCESSES AND NONNEGATIVITY

In [48], (see also [14]) Meyer showed how the group inverse (a special case of the Drazin inverse) could be used to simplify the study of Markov chains. This approach was applied to error estimates in [49], [51]. New results may be found in Meyer's paper [[50]].

With the increased study and concern with large scale linear systems, iterative procedures based on splittings have once again become important. The group inverse was used by Meyer and Plemmons in [52] to study singular splittings. The paper by Hartwig and Hall [[35]] on Cesaro-Neumann iterations is an extension of this earlier work.

Around this same time the volume by Berman and Plemmons on nonnegative matrices [8] came out. This work, and in particular [8] and [52], has spawned a great deal of very recent work dealing with generalized inverses.

Some has dealt directly with iterative procedures [15], [20], [55], [61]. Many papers have been written about generalized inverses of particular types of matrices; circulant [4], [70] incidence [9], [38], integral [26], boolean [4], banded [7], non negative [34], [37], [39], [58], [66], [67], and polynomial or rational [63], [65]. A recent survey on non negative matrices is given in [67]. Closely related is the idea of extending the theory to matrices over finite fields and rings [23], [[35]], [36], [63]. The Drazin inverse for a matrix over a finite field [36] has been applied to certain Cryptographic systems [43]. The Drazin inverse is only defined for square matrices. Since rectangular systems of differential equations arise, a Drazin inverse for a rectangular matrix has been defined and studied [24].

## 4. NUMERICAL PROBLEMS

Generalized inverses arise in several ways in numerical analysis.

The first, and most obvious question, is how to (and what it means to) compute the different generalized inverses. A closely related problem is how to compute the solutions or expressions involving generalized inverses. The situation is similar to finding $A^{-1}$ and solving $A\underline{x} = \underline{b}$ for nonsingular A. Both problems involve similar operations but one would not normally solve $A\underline{x} = \underline{b}$ by computing $A^{-1}$ and then $A^{-1}\underline{b}$. In this volume Wilkinson [[69]] discusses both types of problems in relation to the Drazin inverse and solutions of linear systems of differential equations. The comparable results for the Moore-Penrose and (1) - inverses in the finite dimensional case were known prior to 1976. Some work is still underway [26], [40]. See [64] for a good summary. Generalized inverses can also be used to study the behavior of ill conditioned systems.

Generalized inverses can be part of the procedure for numerically solving a problem. One example was the singular splittings discussed in Section 3, (see [55] for example). Another is Tanabe's paper [[68]] in this volume which shows how in some minimization problems even if the appropriate Jacobian is singular, a generalized inverse can still often be used in a Newton type procedure. See also [27]. As to be expected, in the infinite dimensional case the situation is sometimes more complicated but similar developments exist.

## 5. ELECTRICAL NETWORKS

Of course the work mentioned in Section 2 may be applied to electrical networks. As noted in [17], certain important stabilizable and reachable subspaces for singular systems of the form (1) can be described using the Drazin inverse.

One of the more original applications involving generalized inverses has been the work of Trapp, Anderson, Morley and Duffin in network connections and shorted operators. Some of their early results may be found in [14]. The papers [[2]], [[29]] extend and unify this earlier work.

## 6. INFINITE DIMENSIONAL PROBLEMS

The study of generalized inverses in infinite dimensional spaces, as to be expected, has several different aspects.

First one needs to distinguish between infinite matrices and linear operators on some type of topological vector space. Infinite matrices, for which multiplication can be nonassociative, are considered in [17, Chapter VII] and [[18]]. Most of the literature, however, has assumed some type of topology, and in fact, has tended to be in Hilbert (complete, inner-product), or Banach (complete normed linear) spaces.

There exists a large body of work on defining various types of generalized inverses, determining when they exist, developing their basic properties, and applying them. See, for example [3], [12], [19], [21], [53], [54]. The last five papers in this volume continue these studies.

The spectral decomposition of a Hermitian matrix plays an important role in the theory of the Moore-Penrose inverse and the singular value decomposition. Decomposable operators on Banach spaces represent a generalization of the spectral theory for Hermitian matrices. In this volume, Erdelyi [[31]] studies the generalized inversion of decomposable operators. Campbell [[18]] and Bouldin [[11]] discuss some of the problems of extending the theory of [14], [16], [17] to infinite systems of differential equations and denumberable Markov chains.

Another important question is the calculation of specific generalized inverses both to serve as examples and applications of the theory. Differential operators are still studied [30], [44] and have been recently applied to bifurcation theory [44]. In this volume, Groetsch and Jacobs [[32]] develop an iterative method to compute generalized inverses and Chang [[22]]

applies the theory of generalized inverses to interpolation theory.

Much of the work in the papers of Anderson and Trapp [[2]] and Duffin and Morley [[29]] mentioned earlier can be placed in an infinite dimensional setting.

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# 2 Applications to networks and Markov chains

W N ANDERSON AND G E TRAPP

# Analytic operator functions and electrical networks

## 1. INTRODUCTION

One important application of operator valued analytic functions is to the study of electrical networks; in this tutorial paper we will discuss some aspects of this application. Among the standard references for the results presented here are the textbooks of Hazony [14] and Newcomb [16]. Our particular treatment is based on the papers of Anderson, Duffin and Trapp [1], Bott and Duffin [7] and Duffin [12]. Our notation does not follow the electrical network theory standard.

In Section 2 we discuss the concept of a positive real operator, the type of operator function which arises naturally in studying electrical networks. In Section 3 we discuss the algebraic setting for the study of electrical networks. In Section 4 we show how n-port electrical networks can be analyzed using the techniques previously developed. In the final section we discuss some other aspects of the theory.

In Sections 2, 3, and 4 we will be dealing with linear operators defined on finite dimensional complex Hilbert spaces; inner products will be denoted by $\langle \cdot, \cdot \rangle$. We will let $E$ and $P$ denote respectively $m$ and $n$ dimensional spaces. We assume that $E$ has a fixed orthonormal basis $\{e_1, \ldots, e_m\}$, whose members are called edges, and that $P$ has a fixed orthonormal basis $\{p_1, \ldots, p_n\}$, whose members are called ports. All matrices will be written with respect to these bases. The letter $\lambda$ will denote a complex number; the letter $\omega$ will always be used for a real number. For a linear operator $A$ the adjoint operator $A^*$ is defined by $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all vectors $x$ and $y$. If $A = A^*$, then we say that $A$ is Hermitian. If $A$ is Hermitian, then we say that $A$ is positive semidefinite if $\langle Ax, x \rangle \geq 0$ for all vectors $x$. For Hermitian operators $A$ and $B$, we write $A \geq B$ if $A - B$ is positive semidefinite.

## 2. POSITIVE REAL FUNCTIONS

Among the most important operator valued analytic functions are the