FINITE MATHEMATICS and CALCULUS

with applications

G.M. STRATOPOULOS

FINITE MATHEMATICS AND CALCULUS WITH APPLICATIONS

G.M. STRATOPOULOS
United States International University, San Diego

IEP, A Dun-Donnelley Publisher, New York

PREFACE

I began this book with the conviction that most students need mathematics. Whether they will be professionally concerned with the behavioral, management, or social sciences, or will spend their working lives in administrative or corporate offices, they will inevitably be involved with mathematical applications and will, ideally, realize the pleasurable gains in productivity that come with the mastery of certain techniques, and the awareness of others.

Mathematics is intellectually stimulating, aesthetically pleasing, and certainly useful. Some of the newer applied areas as well as many of those developed and refined over many years are applicable in an amazing range of circumstances. So I have brought together, and tried to order logically, elements of optimization theory, the quantitative underpinnings of management science, operations research, econometrics, linear algebra, and linear programming. I have presented game, sampling, and probability theories as essential tools in the decision-making process. The content of the book became, inevitably, an introduction to topics and applications in finite mathematics and calculus for the behavioral, management, and social sciences. As it turns out, this content is also very suitable for those taking mathematics as a liberal arts requirement. Given the backgrounds and interests of my students, the level of presentation is quite elementary but always related to real world situations, and dependent only on some familiarity with intermediate algebra.

Proofs of theorems are in general omitted. Instead, examples are extensively used to demonstrate the validity of theorems and corollaries. Frequently, illustrations from real situations serve to introduce and clarify certain mathematical notions. In addition, the substantial number of examples and exercises in each section demonstrate both mathematical principles and their relevancy. Answers to selected problems are provided within each exercise. For further discussions of the topics in the book and for proofs of certain theorems and corollaries, we provide references for supplementary readings at the end of each chapter. Throughout the text, "footnote numbering" is used to identify the appropriate reference in the listing of supplementary readings at the end of each chapter.

The table of contents describes the material that can adequately be covered in two semesters or in three quarters either as a prerequisite course or as a liberal arts requirement in mathematics. To provide flexibility in the selection and sequence of content coverage many chapters are written independently of others. For instance, one instructor may choose to introduce the calculus section after presenting Chapters 1 and 2. Another may elect to study only the chapters on determinants and matrices and their applications on (a) linear systems of equations and inequalities, (b) linear transformations and the characteristic value problem, (c) Markov chains, and (d) linear programming and game theory without any loss of continuity.

I would like to express my appreciation to the following people who assisted me in the selection and organization of the materials in this manuscript: Dr. Phillip Beukema, professor of business and economics; Dr. David Jacobs, professor of psychology; Dr. James Gehrmann, professor of mathematics; my secretary, Phyllis Smart, for being such a beautiful typist; and my wife, Dr. Irene C. Stratopoulos, professor of English, for her very substantial contributions to the book.

CONTENTS

PART ONE

Finite Mathematics

CHAPTER 1

Theory of Sets

1.1 Introduction

The construction of mathematical models essential to the analysis and solution of problems in the behavioral, social, management, and physical sciences requires a certain amount of technical vocabulary, some understanding of the laws of logic,[1] and certainly the assumption of certain statements referred to as either axioms or postulates. Thus we shall begin the development of finite mathematics and calculus with the assumption that the student has been introduced to the basic characteristics of mathematics in a high school curriculum and that he has taken at least an intermediate algebra course. Moreover, since the language and techniques of set theory provide not only the necessary technical vocabulary but often the basic tools for the construction of appropriate mathematical models, we begin with an introduction to the fundamentals of set theory. Its extensive and convenient application will become increasingly apparent as we consider the study of combinatorial analysis, sampling theory, and theory of probability as well as other fundamental topics in finite mathematics and calculus. The theory of sets was created and first used by the great mathematician Georj Cantor (1845–1918), who was born in Russia, educated in Germany, and taught there during the years 1874–1895 when his works on this subject were published.

1.2 Definitions and Symbols

A set is simply a collection, class, family, or aggregate of distinct objects that are called "elements" or "members of the set." The elements of a set may be people, points, lines, numbers, or mental concepts. So we may be speaking of the set of counting (natural, positive integers) numbers $1, 2, 3, \ldots,$ or the set of people with IQ's less than 100, or the set of all persons under welfare in the state of California, or even the set of all banks that pay 7% interest on all savings accounts. From the definition of a set it is clear that there exist two types of sets: namely, those with a finite number of elements, on the one hand; and those with an infinite number of elements, on the other.

DEFINITION 1.1 We say that a set is finite if the number of its elements can be represented by a certain natural number n. Otherwise we say the set is infinite.

For example, the set of all natural numbers is infinite, and the set of all people with IQ's less than 100 is finite.

It is customary to denote sets by means of capital letters, and the elements of sets by small letters. We write $a \in A$ to denote that "a is a member of the set A," or to say that "a belongs to A." Similarly, we write $a \notin A$ to denote that "a does not belong to A," or "a is not an element of set A."

A set may be specified in the following two ways:

i. By listing all its elements within braces.
ii. By stating a characteristic property or properties that determine whether or not a given object is an element of that set.

The notation usually adopted for the second method comprises two parts, separated by a vertical line, within braces: The first part tells us what type of element is being considered and the second part specifies the characteristic property. For example, if N is the set of all natural numbers and if S is the set of all those natural numbers whose square is less than 12, then S can be specified as follows:

$$S = \{1, 2, 3\}$$

or

$$S = \{x \in N \mid x^2 < 12\}$$

The latter is read as "S is the set of all natural numbers x for which x^2 is less than 12."

The empty or void set denoted by $\emptyset$ is the set that contains no elements at all. Although the empty set may seem at first to be an artificial notion, its usefulness and acceptance as a bona fide set become increasingly apparent in the study of the algebra of sets.[2] In such an algebra the empty set plays the role (as well as other roles) comparable to the zero number in the algebra of real or complex numbers.

DEFINITION 1.2 Two sets A and B are said to be equal, written $A = B$, if and only if they contain the same elements.

In terms of the membership relation the above definition may be expressed as $A = B$ means "$x \in A$ if and only if $x \in B$." For example, if

$$A = \{1, 2, 3\}$$

if

$$B = \{1, 2, 3, 4\}$$

and if

$$C = \{x \in N \mid x^2 < 13\}$$

then the following are true:

a. $A \neq B$ since $4 \in B$ but $4 \notin A$.
b. $A = C$ since $C = \{x \in N \mid x^2 < 13\}$ implies that

$$C = \{1, 2, 3\}$$

and

$$A = \{1, 2, 3\}$$

The previous example indicates that every element of the set A is an element of the set B, but A is not the same as the set B. The idea illustrated in this example is generalized in Definition 1.3.

DEFINITION 1.3 A set A is said to be a subset of a set B, written $A \subseteq B$, if and only if every element of A is an element of B; A is said to be a proper subset of B, written $A \subset B$, if and only if $A \subseteq B$ but $A \neq B$. Thus $A \subseteq B$ means "if $x \in A$, then $x \in B$." For example, if

$$A = \{x \in N \mid x^3 \leq 27\} \qquad \text{and} \qquad B = \{x \in N \mid x^3 \leq 64\}$$

then

$$A \subseteq B \qquad \text{and} \qquad A \neq B$$

since

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 3, 4\}$$

and every element of A is in B, whereas the number 4 is in B but not in A. Thus A is a proper subset of B. The subset relationship $A \subseteq B$ is often written as $B \supseteq A$.

Theorem 1.1

a. Every set is a subset of itself.
b. The empty set is a subset of every set.
c. If A and B are any two sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

In examining different relationships among sets, let us suppose we have the following three sets:

1. The set B of all independent banks in the state of California.
2. The set P of all presidents of the independent California banks.
3. The set E of all employees of the independent California banks.

It is clear that to each element b in B there corresponds one and only one element p in P, and conversely to each element of P there corresponds one and only one element of B. However, this type of correspondence does not hold true between the sets B and E since the Bank of America in California has more than one employee. This type of relation between two sets is generalized in Definition 1.4.

DEFINITION 1.4 Two sets A and B are said to be in a one-to-one correspondence if and only if their elements can be paired in such a way that each element of A is paired with one and only one element of B, and conversely. Such sets are said to be equivalent, and the equivalence between them is specified by writing $A \sim B$.

In view of this definition we say that the sets B and P defined above are equivalent or that there can be established a one-to-one correspondence between them. On the other hand the sets B and E are obviously not equivalent. Another example of set equivalence is provided by the set M of all married males and the set F of all married females in a society that allows no polygamy for either sex. The necessary one-to-one correspondence for M and F is provided by the marriage license. Yet another example of such a one-to-one correspondence between two infinite sets, in this case, can be established between the set E of all even numbers, and the set O of all odd numbers. (How?)

DEFINITION 1.5 We say that a set of numbers S is bounded if and only if there exist numbers m and M such that $m \leq x \leq M$ for all $x \in S$. The numbers m and M are called "lower" and "upper" bounds of S, respectively. Otherwise we say the set is unbounded. Moreover, we say that S is bounded above (or below) if and only if it has an upper (or lower) bound exclusively. The different bounds of a set may or may not be members of the set.

For example, let

$$A = \{x, \text{real numbers} \mid x > 20\}$$

$$B = \{x, \text{real numbers} \mid x < 0\}$$

and

$$C = \{x \in N \mid 1 \leq x \leq 10\}$$

Then A is bounded below, and a lower bound is $m = 20$; B is bounded above, and upper bound is $M = 0$; and C is bounded with a lower bound $m = 1$ and an upper bound $M = 10$. On the other hand the set of all real numbers is unbounded. (Why?)

DEFINITION 1.6 By a variable we mean a symbol representing an unspecified element of a given set containing more than one element. A symbol representing the element of a set that contains only one member is called a "constant."

For example, if

$$A = \{1, 2, 7\} \qquad \text{and} \qquad B = \{5\}$$

then $x \in A$ implies x is a variable, and $y \in B$ implies y is a constant.

Exercises

Whenever possible, designate each of the sets in Exercises 1 through 8 in two ways.

1. The natural numbers between 3 and 9.
2. The even numbers between 2 and 21.
3. The set of stocks listed on the New York Stock Exchange.
4. The set of all real numbers satisfying each of the equations
   a. $6x^2 + x - 2 = 0$.
   b. $5x^2 + 3x + 7 = 0$.

5. The set of all possible outcomes in the game of tossing and matching pennies by two players.

Ans. Four in number

6. The set of possible stimulus configurations available in an experiment on concept formation in small children in which the experimenter wishes to vary the dimensions of color, size, and number of petals in a flower sketch whenever there are

a. Two colors, two sizes, and two petal configurations.
b. Three colors, three sizes, and three petal configurations.

Ans. (b) 27

7. The set of possible scores of a person required to take three different tests T_1, T_2, and T_3 in each of the two different areas E_1 and E_2.

Ans. Six in number

8. The set of all numbers x such that

a. $2^x > 4$.
b. $2^x < 4$.
c. $1/16 \leq 2^x \leq 64$.

Ans. $-4 \leq x \leq 6$

9. About the previous eight exercises determine the following.

a. All the finite sets.
b. All the infinite sets.
c. All the bounded sets.
d. All the sets bounded above or below only.
e. All the unbounded sets.

10. If $A = \{a, b, c\}$, then list all the subsets of A.

Ans. Eight

11. If a psychiatrist wishes to meet individually and in all possible groups of twos and threes a group of four of his patients, how many different meetings must he schedule?

Ans. 14

12. Let the sets U, A, B, and C be defined as follows:

$U = \{1, 2, 3, 4, 5, 6\}$.
$A = \{1, 2, 3, 4\}$.
$B = \{4, 5, 6\}$.
$C = \{4, 5\}$.

Then, replace the comma with $\subseteq$ or $\nsubseteq$ in each of the following:

a. A, U. b. C, A. c. A, B. d. C, B.

13. About the sets:

$N = \{\text{all natural numbers}\}$.
$A = \{\text{all even natural numbers}\}$.
$B = \{\text{all odd natural numbers}\}$.
$C = \{x \mid 1 < x < 5\}$.
$D = \{x \mid x < 9\}$.

Determine which of the following are true?

a. $A \sim D$	b. $C = D$	c. $C \subset D$
d. $A \sim B$	e. $A = B$	f. $D \subset C$
g. $A \sim N$	h. $A \subset B$	i. $A \subset N$
j. $C \sim D$	k. $C \subset A$	l. $C \subset B$.

14. State all the possible one-to-one correspondences between the sets

$A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$

Ans. 24 in number

1.3 The Algebra of Sets

In practice it is often useful as well as very convenient to adopt geometric language for sets. For instance, we may choose to call "point" an "element of a set" even though the element has no obvious geometric character. When we adopt such descriptive language, we must bear in mind that we do so only for convenience and must carefully refrain from assuming any properties not legitimately established.

Such a geometric interpretation of sets suggests the use of sketches, called "Venn diagrams," to represent sets and relations between sets. Thus the subset relation $A \subset B$ can be shown graphically as in Figure 1.1.

John Venn (1834–1923), the great English logician and ordained priest, who resigned his order in 1883 to devote all his time to the study and teaching of logic, first employed these diagrams in his work *Symbolic Logic*, published in 1881.

These diagrams provide insight concerning sets, and often suggest methods by which statements about sets can be proved or disproved. However, diagrams are not valid substitutes for formal proofs.

With these preliminaries in mind, let us now consider a set U, called the "universal set," together with the set K consisting of all the subsets of U, and let us define the following important operations on the set K, that is, on the subsets of U.