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Computational Group Theory and the Theory of Groups, II

Computational Group Theory and Cohomology
August 4–8, 2008
Harlaxton College, Grantham, United Kingdom

AMS Special Session
Computational Group Theory
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The photograph of Harlaxton College appears courtesy of the University of Evansville

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Preface

This is a follow-up to the Contemporary Mathematics volume that we edited on the same topic and that was published by the American Mathematical Society in 2008. This volume consists of contributions by researchers who were invited to the Harlaxton Conference on Computational Group Theory and Cohomology, held at Harlaxton College, The British Campus of the University of Evansville, August 4-8, 2008; and to the AMS Special Session on Computational Group Theory held at Western Michigan University, October 17-19, 2008. The Harlaxton Conference was supported financially by the De Brun Centre for Computational Algebra, National University of Ireland, Galway, and the University of Evansville's Institute for Global Enterprise in Indiana.

Both the Conference and the Special Session focused on examples of using CGT to solve problems that arise from many areas of group theory; in this volume we find applications to the enumeration of subgroups of the symmetric group, covering groups by subgroups, the ongoing co-class project for classification of p-groups, construction or computation of homological and cohomological invariants of groups, probabilistic group theory, and the study of free groups, among others. Computational Group Theory plays many roles in these investigations, from exploration that suggests conjectures or proofs, through performing key computations required to establish theorems.

Once again, we present these examples in the hope that they will encourage researchers and graduate students to think about ways in which they can incorporate CGT in their own research by seeing many different applications of CGT to traditional problems in Group Theory.

The Harlaxton Conference was organized by Bettina Eick, Graham Ellis, and Robert F. Morse; we thank them very much for all their work, and we also thank the de Brun Centre and the University of Evansville for their financial support.

The second editor was supported in part by a grant from the Louisiana Board of Regents. The first and third editors thank Arturo for his work managing and editing the submissions. The three of us are grateful to all the participants in the conferences, and to all authors who submitted contributions to this volume. We are also very thankful indeed to the referees who did such an excellent and timely job for both this and the previous volume. Finally, we are also very grateful to the American Mathematical Society for their help in the publication of this volume, particularly to Christine M. Thivierge for her help and patience.

Luise-Charlotte Kappe
Arturo Magidin
Robert Fitzgerald Morse

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The Probabilistic Zeta Function

Bret Benesh

ABSTRACT. This paper is a summary of results on the $P_G(s)$ function, which is the reciprocal of the probabilistic zeta function for finite groups. This function gives the probability that s randomly chosen elements generate a group G, and information about the structure of the group G is also embedded in it.

1. Introduction and History

Probabilistic group theory has been a growing field of mathematics for the past couple of decades. While other papers have considered this field in greater generality (see [**Di, Shal1, Shal2**]), we will be focusing on the so-called $P_G(s)$ function, which is the function that gives the probability that s randomly chosen elements (with replacement) of a finite group G generate G.

The study of the $P_G(s)$ function began in 1936, when Philip Hall [**H**] created the Eulerian function $\phi_G(s)$, defined to be the number of s-tuples $(g_1, \ldots, g_s) \in G^s$ such that $\langle g_1, \ldots, g_s \rangle = G$. Hall showed that

$$\phi_G(s) = \sum_{H \leq G} \mu_G(H)|H|^s,$$

where $\mu_G(H)$ is the Möbius function of the subgroup lattice of G, defined inductively as $\mu_G(G) = 1$ and

$$\sum_{H \leq K \leq G} \mu_G(K) = 0$$

if $H < G$.

After Hall, G.E. Wall [**W**] used a variation of the Eulerian function (the Eulerian polynomial) to prove the following theorem.

THEOREM 1.1 (Wall's Theorem). *If G is a finite solvable group, then the number of maximal subgroups in G is less than $|G|$.*

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Key words and phrases. group theory, zeta functions, Dirichlet series, subgroup lattices, Moebius functions.

The author would like to thank Nigel Boston, Erika Damian, and the referee for their thoughtful comments on the paper.

Wall also conjectured that this result holds for nonsolvable groups, and relevant work has been done to that end in [**LiSh1**] and [**LiPySh**].

The most recent wave of interest in this field began in 1996, when Nigel Boston [**Bo**] and Avinoam Mann [**Ma**] independently defined the $P_G(s)$ function described above. It is clear that $P_G(s) = \frac{\phi_G(s)}{|G|^s}$, or

$$P_G(s) = \sum_{H \le G} \frac{\mu_G(H)}{|G:H|^s}$$

by using Hall's result, and thus $P_G(s)$ is a Dirichlet series.

A word on the motivation behind this paper: while this topic is interesting in its own right – the author wrote his thesis [**Be**] on a similar subject – it is also a viable topic for undergraduate research. The basic idea is accessible, as only some knowledge of groups and proportions are needed. While many of the ideas below are too advanced for most undergraduates, several of them are not; Boston made a conjecture about the derivative of $P_G(s)$, soon solved by Shareshian in [**Shar**], that a calculus student could understand. Moreover, this topic lends itself well to computational algebra packages like GAP [**G**] and MAGMA [**BoCaPl**]. In fact, Boston's 1996 paper references the use of Cayley, an early version of Magma. Use of a computational algebra package would reduce the amount of background knowledge needed for an undergraduate to begin research, as the student could use simple programs to make conjectures about the $P_G(s)$ function.

2. The Basics of $P_G(s)$

We begin with some basic facts about $P_G(s)$, which are largely from [**Bo**] and [**Ma**]. First, several examples of $P_G(s)$, courtesy of Boston:

Cyclic Groups C_n: $P_{C_n}(s) = \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p^s}\right)$.

The Alternating Group A_4: $P_{A_4}(s) = \left(1 - \frac{2}{2^s}\right)\left(1 + \frac{2}{2^s}\right)\left(1 - \frac{1}{3^s}\right)$.

The Alternating Group A_5: $P_{A_5}(s) = 1 - \frac{5}{5^s} - \frac{6}{6^s} - \frac{10}{10^s} + \frac{20}{20^s} + \frac{60}{30^s} - \frac{60}{60^s}$.

S^n for a Simple Group S: $P_{S^n}(s) = \prod_{i=0}^{n-1} \left(P_S(s) - \frac{i|\text{Aut } S|}{|S|^s}\right)$.

By way of motivation, the probability that two integers chosen at random are relatively prime can be solved, rather non-rigorously, by

$$\prod_{\text{primes } p} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

where $\zeta(s)$ is the Riemann zeta function. The left side of the above equation resembles a product of $P_{C_p}(s)$ functions evaluated at $s = 2$ and, we can think of $\frac{1}{P_G(s)}$ as a zeta function of G. We define $\frac{1}{P_G(s)}$ to be the *probabilistic zeta function,* and it is common to label results for $P_G(s)$ as results about the probabilistic zeta function. While it initially only makes sense to consider natural numbers s in the function $P_G(s)$, we will see below that we can gain insight into G by expanding the domain to the complex numbers.

The ring of finite Dirichlet series with coefficients in $\mathbb{Z}$ is a unique factorization domain, so factoring $P_G(s)$ will be of great interest. In fact, if N is a normal subgroup of G, then we can factor $P_G(s)$ as

$$P_G(s) = P_{G/N}(s)P_{G,N}(s),$$

where $P_{G,N}(s)$ is given by the formula

$$P_{G,N}(s) = \sum_{\substack{H \leq G \\ HN=G}} \frac{\mu_G(H)}{|G:H|^s}$$

and interpreted as the conditional probability that a random s-tuple $(g_1, \ldots, g_s)$ of G^s generates G given that $\langle g_1, \ldots, g_s, N\rangle = G$.

While one might conjecture that $P_{G,N}(s) = P_N(s)$, this is only sometimes true ([**Bo**]). Consider the symmetric group S_5 and its alternating group A_5. It is true that $P_{S_5}(s) = P_{C_2}(s)P_{A_5}(s)$, so that $P_{A_5}(s) = P_{S_5,A_5}(s)$. However, we have

$$P_{S_3}(s) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{3}{3^s}\right),$$

so $P_{S_3,C_3}(s) = 1 - \frac{3}{3^s} \neq 1 - \frac{1}{3^s} = P_{C_3}(s)$.

Recall that a chief series

$$1 = N_0 \subset N_1 \subset \cdots \subset N_n = G$$

is a collection of normal subgroups N_i of G such that N_{i+1}/N_i is minimal normal in G/N_i. Then the factorization $P_G(s) = P_{G/N}(s)P_{G,N}(s)$ can be used repeatedly on a chief series to obtain factors of $P_G(s)$. Detomi and Lucchini [**DeLu1**] proved that the factorization is independent of the choice of chief series (Gaschütz [**Ga**] had previously proved this independence for solvable groups).

Another immediate result from the above factorization is that if $P_G(s)$ is irreducible, then G is simple. The converse is not true, however. For instance,

$$P_{PSL(2,7)}(s) = \left(1 - \frac{2}{2^s}\right)\left(1 + \frac{2}{2^s} + \frac{4}{4^s} - \frac{14}{7^s} - \frac{28}{14^s} + \frac{21}{21^s} - \frac{28}{28^s} + \frac{42}{42^s}\right)$$

is reducible. In fact, $P_{PSL(2,p)}(s)$ is always reducible when $t \equiv 3 \pmod 4$ and $p = 2^t - 1$ ([**DamLuMo**]).

The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all maximal subgroups and equals the set of non-generators. If we have a normal subgroup N contained in the Frattini subgroup, N contains only non-generators and we obtain $P_G(s) = P_{G/N}(s)$, since $P_{G,N}(s) = 1$.

3. A Motivating Application

An important application of the $P_G(s)$ function is to help determine the minimal number of generators of a group H, denoted $d(H)$. For example, let H be a finite group such that $d(H) = m + 1$ for some m, but all proper quotients Q of H have the property that $d(Q) \leq m$. Dalla Volta and Lucchini [**DaLu**] proved such an H must be isomorphic to

$$L_t = \{(l_1, \ldots, l_t) \in L^t \mid l_1 \equiv \cdots \equiv l_t \pmod M\},$$

where L is a group with unique minimal normal subgroup M (the group L is called a *primitive monolithic group*) and t is some integer.

The probabilistic zeta function pops up in determining what the integer t is. In the same paper, Dalla Volta and Lucchini proved that if M is nonabelian, then

$$t = 1 + \frac{|M|^m P_L(m)}{|C_{\mathrm{Aut}L}(L/M)| P_{L/M}(m)} = 1 + \frac{|M|^m}{|C_{\mathrm{Aut}L}(L/M)|} P_{L,M}(m).$$

The group L_t is useful in proofs by contradiction that consider minimal counterexamples, as then $P_{L,M}(s)$ can be useful in proving that minimal counterexamples do not exist.

4. What $P_G(s)$ Says About G

As evidenced by the formula

$$P_G(s) = \sum_{H \le G} \frac{\mu_G(H)}{|G:H|^s},$$

$P_G(s)$ is tied to the subgroup structure of the group G. Because of this, one can think of it as encoding information about the structure of G. This section focuses on examples of information one can gain about a group G solely from knowing the $P_G(s)$ function.

4.1. The primes dividing $|G|$. The first piece of data we can get from $P_G(s)$ is exactly which primes divide $|G|$. Damian and Lucchini [**DamLu1**] proved that if $P_G(s)$ is written as $\sum \frac{a_n}{n^s}$, then:

THEOREM 4.1. *A prime p divides $|G|$ if and only if p divides n for some n with $a_n \neq 0$.*

4.2. The coset poset. Our second example is a case where it is advantageous to view $P_G(s)$ as having a domain greater than the non-negative integers. Brown and Bouc [**Br**] found that letting $s = -1$ gives interesting topological information about the group G. The coset poset $\mathcal{C}(G)$ is the set of cosets xH ($x \in G$ and $H < G$) ordered by inclusion. We can use a simplicial complex $\Delta(\mathcal{C}(G))$ whose simplices are the finite chains in $\mathcal{C}(G)$ to define the Euler characteristic $\chi(\mathcal{C}(G))$. We may then define the reduced Euler characteristic $\tilde{\chi}(\mathcal{C}(G)) = \chi(\mathcal{C}(G)) - 1$. Then Bouc discovered:

THEOREM 4.2. $P_G(-1) = -\tilde{\chi}(\mathcal{C}(G))$.

Moreover, Brown defined an analogue of $P_G(s)$ for finite lattices (instead of groups). Using this analogue, Brown shows that the entire $P_G(s)$ function, not only its value at $s = -1$, can be recreated from a lattice that is defined from the coset poset $\mathcal{C}(G)$.

4.3. Solvability, supersolvability, and nilpotency. Since $P_G(s)$ encodes information about the structure of G, it is natural to wonder whether solvability questions can be answered based solely on $P_G(s)$. Gaschütz [**Ga**] began working on this question in the 1950s, and this question was completely answered by Detomi and Lucchini [**DeLu2**] with the following theorem:

THEOREM 4.3. *G is solvable if and only if $P_G(s)$ is a product of factors of the form $1 - \frac{c_i}{(p_i^{n_i})^s}$, where p_i is a prime.*

A group is supersolvable if it has an invariant normal series where all factors are cyclic. Detomi and Lucchini also describe a condition for supersolvable groups.

THEOREM 4.4. *G is supersolvable if and only if $P_G(s)$ is a product of factors of the form $1 - \frac{c_i}{p_i^s}$, where each p_i is prime and each c_i is positive.*

This begs the question of whether a similar result exists for nilpotent groups, but Gaschütz demonstrated that no such result can exist. Indeed, the functions $P_G(s)$ for $G = C_2 \times C_3 \times C_3$ (nilpotent) and for $G = S_3 \times C_3$ (solvable, but not nilpotent) are both equal to

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{3}{3^s}\right).$$

Therefore, it is impossible to determine nilpotency strictly from the $P_G(s)$ function. However, Damian and Lucchini [**DamLu2**] did find the following result on nilpotency. First, define

$$P_G(H, s) = \sum_{H \leq K \leq G} \frac{\mu_G(K)}{|G : K|^s}.$$

Then:

THEOREM 4.5. *A group G is nilpotent if and only if $P_G(H, s)$ divides $P_G(s)$ for all $H \leq G$.*

Finally, suppose that $P_G(s) = \sum \frac{a_n}{n^s}$. Then Detomi and Lucchini [**DeLu2**] proved that G is solvable if and only if $a_{nm} = a_n a_m$ whenever $(n, m) = 1$. Damian and Lucchini [**DamLu1**] were able to generalize this to p-solvable groups:

THEOREM 4.6. *Suppose $P_G(s) = \sum \frac{a_n}{n^s}$. Then G is p-solvable if and only if $a_{p^r d} = a_{p^r} a_d$ whenever $(p, d) = 1$.*

4.4. Simple groups. We now turn our attention to nonsolvable groups, the results of which are found in [**DamLu3, DamLu4, DamLuMo**]. All three papers work toward the same result, which culminates in the following theorem:

THEOREM 4.7. *Let G be a nonabelian finite simple group, let H be a finite group with trivial Frattini subgroup, and assume $P_G(s) = P_H(s)$.*

(1) *If G is an alternating group or a sporadic simple group, then $G \cong H$.*
(2) *If G and H are groups of Lie type defined on a field of characteristic p, then $G \cong H$.*

Finally, Nigel Boston [**Bo**] conjectured that $P'_G(1) = 0$ whenever G is simple nonabelian. This conjecture was proved and generalized by John Shareshian [**Shar**] in the following theorem.

THEOREM 4.8. *$P'_G(1) = 0$ unless $G/O_p(G)$ is cyclic for some prime p.*

5. Computing $P_G(s)$

While $P_G(s)$ can be tedious to compute by hand, computer algebra systems such as MAGMA and GAP can quickly generate the formula and numerical values for $P_G(s)$. The key to computing such a function for a group G is knowing its subgroups, their indices, and the Möbius function. These can all be obtained from MAGMA and GAP, although the subgroups are typically given as conjugacy classes. Because of this, the length of each conjugacy class (that is, the number of subgroups in each class) is also required.

The computer algebra system GAP offers a convenient shortcut for computing examples of the $P_G(s)$ function: it has a command to compute the table of marks. Briefly, the table of marks of a group is a matrix whose entries describe the number of fixed points when a representative of one conjugacy class of subgroups of G acts on another via conjugation. The Möbius function can be determined from the table of marks, as described by Pfeiffer in [**Pf**].

GAP makes it very easy to access this information from the table of marks. In fact, a program—in its entirety—that computes the numerical answer to $P(G,s)$ is:

```
P1:=function(G,s)
return EulerianFunctionByTom(TableOfMarks(G),s)/Order(G)^s;
end;
```

Creating a string that returns the formula is slightly more complicated, although not much more so. Easy access to the Möbius function values, lengths, and orders of the conjugacy classes of subgroups are gotten through the command `TableOfMarks(G)` Below is a very basic program for GAP that returns the formula as a string:

```
P2:=function(G)
local i,tom,mob,ord,len,finalstring;
tom:=TableOfMarks(G);
mob:=MoebiusTom(tom).mu;
ord:=OrdersTom(tom);
len:=LengthsTom(tom);
finalstring:="";

for i in [1..Length(mob)] do
 if IsBound(mob[i]) then
  finalstring:=Concatenation(finalstring,"+",String(len[i]*mob[i]),
   "/",String(Order(G)/ord[i]),"^s);
 fi;
od;

return finalstring;
end;
```

This particular program was designed for simplicity, and the resulting string lacks a certain beauty. Because of this, readers are implored to add additional code to make a more readable output. Additionally, shortcuts can be made for efficiency, such as inserting the line of code `G:=G/FrattiniSubgroup(G);` at the beginning of the program to take advantage of the fact that $P_G(s) = P_{G/\Phi(G)}(s)$.

The logic is similar when using MAGMA; the main difference is that MAGMA does not have a command to access the table of marks, and cannot immediately access the Möbius function of a group. However, the command `SubgroupLattice(G)` contains the lengths and orders of the subgroups of G. Additionally, the command `SubgroupLattice(G)` contains information about the containment of the subgroups, and one can use `SubgroupLattice(G)` and the recursive definition of the Möbius function to create a function that returns the Möbius value of a subgroup.

6. Conjectures and Open Problems

We conclude with several unsolved conjectures and avenues for exploration.

(1) Shareshian [**Shar**] proved that $P'_G(1) = 0$ for simple nonabelian G. It is also true that $P''_{A_6}(1) = 0$. Describe all groups G such that $P''_G(1) = 0$.

(2) Boston [**Bo**] observes that $P_{S_n}(s) = P_{A_n}(s)P_{C_2}(s)$ for $n = 2$, 5, and 6 but not for $n = 3$, 4, 7, 8, and 9. Determine for which n the above equation holds.

(3) A generalization of the previous question: describe the nonabelian finite simple groups S such that $P_S(s) = P_{\mathrm{Aut}\, S, S}(s)$.

(4) If $P_{G,N}(s) \neq P_N(s)$, describe the possibilities for $P_{G,N}(s)$. Detomi and Lucchini [**DeLu1**] gave a partial answer to this challenge in 2003. Let L be a finite group with unique minimal normal subgroup M. Then define the following:
   - $\tilde{P}_{L,1}(s) = P_{L,M}(s)$
   - $\tilde{P}_{L,i}(s) = P_{L,M}(s) - \frac{(1+q_M+\cdots+q_M^{i-2})\gamma_M}{|M|^s}$ if $i > 1$

   where $\gamma_M = |C_{\mathrm{Aut}\, M} L/M|$, $q_M = |\mathrm{End}_L M|$ if M is abelian, and $q_M = 1$ otherwise. Detomi and Lucchini proved that each factor of $P_G(s)$ is equal to $\tilde{P}_{L,i}(s)$ for some primitive monolithic group L and positive integer i, thereby reducing the problem to the study of monolithic groups. Determine the possible values of $P_{L,M}(s)$.

(5) Similar to the earlier result on simple groups, we may conjecture:

   CONJECTURE 6.1. *If G is a simple nonabelian finite group, H a finite group with trivial Frattini subgroup, and $P_G(s) = P_H(s)$, then $G \cong H$.*

   This conjecture would follow if the next conjecture were true. First, some notation. Let

$$a_n(G) = \sum_{n=|G:H|} \mu_G(H),$$

   and let

$$b_n(G,p) = \begin{cases} a_n(G) & \text{if } p \nmid n, \\ 0 & \text{otherwise;} \end{cases}$$

   finally, let $P_G^{(p)}(s) = \sum_{n=1}^{\infty} \frac{b_n(G,p)}{n^s}$. Then:

   CONJECTURE 6.2. *Let G be a group of Lie type. Except for finitely many exceptions, a prime p is the characteristic of the defining field if and only if $\left|P_G^{(p)}(0)\right|$ is a nontrivial p-power.*

   Patassini [**Pat**] has provided some evidence that this conjecture is true.

(6) There have been many theorems of the form "$P_{G,N}(s) > \gamma$ whenever $s \geq f(G)$ for some constant γ and some function f of G" (see for example [**DamLuMo, DeLuMo, LuMo, Pak**]). Improve one of these bounds, or determine similar bounds for $P_G(s)$ (Pak [**Pak**] proves something similar to this).

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Department of Mathematics, College of Saint Benedict/Saint John's University, 37 College Avenue South, Saint Joseph, Minnesota 56374

E-mail address: bbenesh@csbsju.edu

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Periodicities for graphs of p-groups beyond coclass

Bettina Eick and Tobias Rossmann

Abstract. We use computational methods to investigate periodic patterns in the graphs $\mathcal{G}(p,(d,w,o))$ associated with the p-groups of rank d, width w, and obliquity o. In the smallest interesting case $\mathcal{G}(p,(3,2,0))$ we obtain a conjectural description of this graph for all $p \geq 3$; in particular, we conjecture that this graph is virtually periodic for all $p \geq 3$. We also investigate other related infinite graphs.

1. Introduction

Which invariants are useful in the classification of p-groups?

The *order* has been considered in many publications, going back to the beginnings of abstract group theory in the 19th century; see [1] for a history. Nowadays, the p-groups of order dividing 2^9 (see [5]) and p^7 (see [16]) are available, but a full classification of the groups of order p^n in general still seems to be out of reach. An important step towards a full classification would be a proof of the famous PORC conjecture [7] which asserts that for fixed n, the number $f(p)$ of p-groups of order p^n is a polynomial on residue classes.

Leedham-Green and Newman [14] suggested using the *coclass* to classify p-groups. Recall that the coclass of a finite p-group G of order p^n and nilpotency class $\mathrm{cl}(G)$ is defined as $\mathrm{cc}(G) = n - \mathrm{cl}(G)$. A first and fundamental idea in classifying all p-groups of a given coclass r is to visualize them in a graph $\mathcal{G}(p,r)$: the vertices of this graph correspond to the isomorphism types of p-groups of coclass r and there is a directed edge $G \to H$ if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$ holds, where $\gamma_i(H)$ denotes the ith term of the lower central series of H. The classification of all p-groups of coclass r thus translates to an investigation of the infinite graph $\mathcal{G}(p,r)$.

Coclass theory has become a rich and interesting research field in group theory. A highlight in this theory was the complete proof of the coclass-conjectures [14] by Shalev [18] and Leedham-Green [11]. We refer to the book by Leedham-Green and McKay [13] for background and details. Nowadays, the fundamental aim in coclass theory is to prove that every graph $\mathcal{G}(p,r)$ can be constructed from a finite subgraph using certain periodic patterns. This has been proved for $p = 2$ in [3] and [4], but is still open for odd primes. A central problem in the odd prime case is that

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the graphs $\mathcal{G}(p,r)$ are usually rather thick and thus are often difficult to investigate in detail. As a consequence, only very little detailed experimental evidence on the structure of these graphs is available and explicit conjectures on the nature of any useful periodic patterns are vague at present.

Leedham-Green thus suggested to try other invariants with a similar approach as in coclass theory with the hope of obtaining graphs which have all the nice features of the graphs $\mathcal{G}(p,r)$, but are thinner and thus easier to understand. In particular, Leedham-Green initiated the classification of p-groups by *rank, width and obliquity*; see Chapter 12 of [13] for a discussion. We briefly recall the definitions of these invariants: for any finite or infinite pro-p-group G and a closed subgroup H of G, let $[G:H]_p$ denote the p-logarithm of the index $[G:H]$; further let $\mathrm{d}(G) = [G:\Phi(G)]_p$ be the cardinality of a minimal (topological) generating set of G, and let $\mu_i(G)$ denote the intersection of all closed normal subgroups of G which are not properly contained in $\gamma_i(G)$. Then we define for a pro-p-group G:

- its *rank* $\mathrm{r}(G) = \sup\{\mathrm{d}(U) \mid U \text{ a closed subgroup of } G\}$,
- its *width* $\mathrm{w}(G) = \sup\{[\gamma_i(G):\gamma_{i+1}(G)]_p \mid i \in \mathbb{N}\}$, and
- its *obliquity* $\mathrm{o}(G) = \sup\{[\gamma_i(G):\mu_i(G)]_p \mid i \in \mathbb{N}\}$.

The obliquity of a group determines how restricted its lattice of normal subgroups is. In particular, in a group of obliquity 0 every normal subgroup lies between two consecutive terms of the lower central series.

Let $\tau(G)$ denote the triple $(\mathrm{r}(G), \mathrm{w}(G), \mathrm{o}(G))$ and define the graph $\mathcal{G}(p,(d,w,o))$ similar to the coclass graphs: the vertices of this graph correspond to the isomorphism types of finite p-groups G with $\tau(G) = (d,w,o)$ and there is a directed edge $G \to H$ if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$ holds. The classification of all p-groups G with $\tau(G) = (d,w,o)$ now translates to understanding the (usually) infinite graph $\mathcal{G}(p,(d,w,o))$.

In this paper we discuss how computational tools can be used to investigate the graphs $\mathcal{G}(p,(d,w,o))$ and we exhibit experimental results for some small and interesting cases. Thus, we give a conjectural description of the graph $\mathcal{G}(p,(3,2,0))$ for $p > 2$ based on our experimental data. It suggests that $\mathcal{G}(p,(3,2,0))$ can be constructed from a finite subgraph using certain periodic patterns and hence $\mathcal{G}(p,(3,2,0))$ seems to have the nice features displayed by the coclass graphs $\mathcal{G}(2,r)$ and, moreover, it is a rather thin graph which can be easily exhibited.

An interesting family of infinite pro-p-groups G with finite $\tau(G)$ are the Sylow pro-p-subgroups of $\mathrm{Aut}(L)$ for simple Lie algebras L of the type $L = \mathfrak{sl}_n(K)$ for $p \geq 3$, where $K/\mathbb{Q}_p$ is a finite extension. The lower central series quotients of such a group G define an infinite path through the graph $\mathcal{G}(p,\tau(G))$. We show how our computational tools can be used to investigate these infinite paths together with certain branches associated with them. Our experiments with these infinite trees indicate that they also exhibit periodic patterns of the same type as $\mathcal{G}(p,(3,2,0))$.

Throughout this paper we assume that p is an odd prime.

2. Preliminaries

There is a correspondence between the infinite paths in $\mathcal{G}(p,(d,w,o))$ and the isomorphism types of infinite pro-p-groups G with $\tau(G) = (d,w,o)$. Hence a first aim in understanding $\mathcal{G}(p,(d,w,o))$ is a classification of the infinite pro-p-groups G with $\tau(G) = (d,w,o)$. In this section, we recall some basic facts about these groups.