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Annals of Discrete Mathematics (16)

General Editor: Peter L. Hammer

University of Waterloo, Ont., Canada

Bonn Workshop on Combinatorial Optimization

Edited by

Achim BACHEM
Martin GRÖTSCHEL
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Bonn Workshop on Combinatorial Optimization

Based on lectures presented at the IV. Bonn Workshop on Combinatorial Optimization, August 28–30, 1980 organised by the Institute of Operations Research and sponsored by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 21

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PREFACE

The field of combinatorial optimization has experienced a tremendous growth in recent years. This is for instance documented by the publication of many new scientific journals in this area as well as by the considerable number of large international conferences taking place every year.

Big meetings have the advantage of bringing a large number of people together and making a quick exchange of new results possible. Due to the (mostly) hectic atmosphere, however, they do not provide a platform for discussing problems in detail and digging deep into new aspects. This is the purpose of a workshop where few people gather together and even fewer people are given extensive time to present their ideas. Moreover, an informal atmosphere not restricted by time limits makes a more profound discussion of all aspects of the new developments possible.

From August 28 to August 30, 1980 the IV. Bonn Workshop on Combinatorial Optimization was held at the Rheinische Friedrich-Wilhelms-Universität, Bonn. It was organized by the Institut für Ökonometrie und Operations Research and generously sponsored by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 21.

Altogether 54 scientists from 16 different countries gathered at this meeting in a highly stimulating atmosphere. This volume constitutes a part of the outgrowth of the workshop and is based on the lectures presented there. The papers cover a broad spectrum of the field from submodular functions to perfect graphs, and from vertex packing to scheduling and subtree extension. All papers were subjected to a careful refereeing process.

We would like to express our sincere thanks to all authors for their cooperation, to all referees for their outstanding (albeit anonymous) contributions, and to the editor and publishers of this series for their support of this venture.

Bonn, October 1981

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CONTENTS

Annals of Discrete Mathematics 16 (1982) 1–26

PARITY GRAPHS

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A graph $G = (V, E)$ is a parity graph if and only if for every pair of vertices (x, y) of G all the minimal chains joining x and y have the same parity.

A characterization of these graphs can be given by a condition on the odd cycles: parity graphs are just the graphs in which every odd cycle has two crossing chords. A theorem of Sachs states that these graphs are perfect.

These graphs are then studied from the algorithmic viewpoint. Polynomial algorithms are defined to recognize them, and to solve the following problems: maximum independent set, minimum coloring, minimum covering by cliques, maximum clique.

1. Introduction

It is rather strange that, when a class of perfect graphs has been characterized, the algorithmic aspect is seldom studied. In particular, there are not only the classical problems of perfect graphs (finding a maximum stable set and a minimum coloring), but also the major problem of recognizing such a class of graphs in polynomial time.

These problems remain unsolved for many classes of perfect graphs (Meyniel's graphs [11], perfect 3-chromatic graphs [16], perfect planar graphs [15]). The only exception is the general paper of Grötschel, Lovász and Schrijver [7] which gives a polynomial algorithm for maximum weighted independent set and minimum coloring for all perfect graphs. This algorithm based on the ellipsoid method unfortunately gives no idea of the structure of perfect graphs, and at the present time appears to be of no great combinatorial interest.

There exist classes of perfect graphs for which these problems are solved: bipartite graphs and their line graphs, triangulated graphs, comparability graphs, and their complements. The recognition problem is also solved for a number of subclasses of these latter graphs (see [6]).

Finally there are classes of graphs for which these problems are not all solved (for example perfect claw-free graphs ([8, 9, 12]), for which the recognition is not yet settled, to our knowledge).

In this study, we deal with a particular class of perfect graphs which is a fairly natural extension of bipartite graphs: parity graphs.

Notation. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. When no confusion is possible we will write V and E for $V(G)$ and $E(G)$.

Definition 1. A *minimal chain* is an elementary chain which is an induced subgraph.

In the graph of Fig. 1, chains (x, z, t, v, y) and (x, z, u, v, y) are minimal but not, for example, (x, z, t, u, v).

Definition 2. The *parity of a minimal chain* is the parity of the number of its edges.

In particular, if two vertices x and y are adjacent, the only minimal chain joining them is the chain reduced to the edge (x, y); this chain is odd.

Definition 3. A (simple, undirected) graph $G = (V, E)$ is called a *parity graph* if, for every pair of vertices x and y of G, all minimal chains joining x and y have the same parity.

Clearly, the notion of a parity graph generalizes that of a bipartite graph. Cliques are non-bipartite parity graphs. The graph depicted in Fig. 2 is a less trivial example.

In Section 2 we prove that a graph G is a parity graph, if and only if each odd cycle of length at least five contains two crossing chords. A theorem of Sachs [13] enables us to confirm that these graphs are perfect.

In Section 3 we prove some properties of these graphs, and we specify their minimal separating sets. These results can be compared to those of Gallai [5] for o-triangulated graphs.

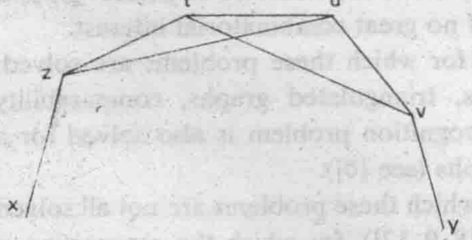

Fig. 1.

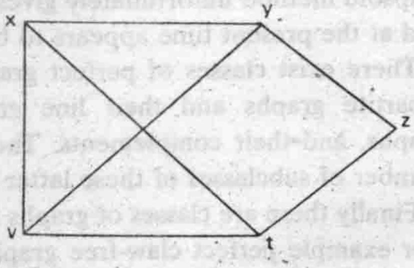

Fig. 2.

In Section 4 a polynomial algorithm for parity graph recognition is given. It is also shown that these graphs are in fact built from two classes of perfect graphs: bipartite graphs, and 'cographs' studied by Corneil, Lerchs and Stewart [3], here called 2-parity graphs.

More precisely: the class of perfect graphs is closed under making true or false twins (that is replacing a vertex by a set of two vertices linked or not by an edge) [10] and under certain extensions by bipartite graphs (cf. Definition 14). It will follow that parity graphs are exactly those graphs arising by these operations from a single point.

Finally, in Section 5, polynomial algorithms are defined for the four above-mentioned problems (in cardinality and in weight). This will yield another proof of the fact that these graphs are perfect.

2. Characterization

Definition 4. We say that *two chords* (x, y) and (z, t) of an elementary cycle *cross*, if the vertices x, z, y, t are different and in this order on the cycle.

Theorem 1. *A graph $G = (V, E)$ is a parity graph, if and only if every odd elementary cycle has two crossing chords.*

Proof. *Necessary condition*: The condition is necessary for a cycle of 5.

Suppose we admit the property on an odd cycle of cardinality k ($k > 5$) and prove it is still true for a cycle of cardinality $k + 2$.

It is easy to check that such an odd cycle contains at least two chords, and two chords which do not cross create at least one odd cycle, whose cardinality is lower than or equal to k, and the property follows by induction.

Sufficient condition: Take a graph G which verifies the condition and which is not a parity graph. As the structure we want for G is hereditary (under taking induced subgraphs) we shall choose a counter-example which is minimal with respect to the vertices.

This counter-example has two vertices x and y joined by an even minimal chain $(x, u_1, \ldots, u_t, y)$ and an odd minimal chain $(x, v_1, \ldots, v_s, y)$ with $s \geq 2$ (cf. Fig. 3).

Let

$$i_0 = \min\{i \mid \exists j > 1: (u_i, v_j) \in E\},$$

$$j_0 = \max\{j \mid (u_{i_0}, v_j) \in E\}.$$

So $i_0 < t$ and $j_0 > 1$, since the odd cycle formed by these two chains has two

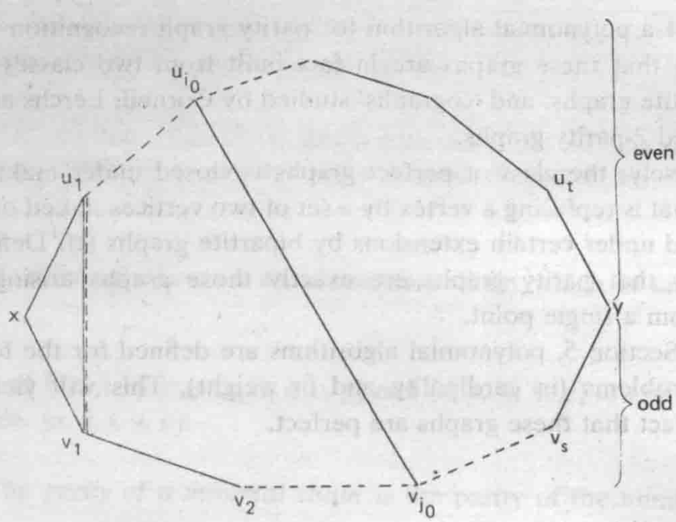

Fig. 3.

crossing chords by assumption. Now $(u_{i_0}, v_{j_0}, v_{j_0+1}, \ldots, v_s, y)$ and $(u_{i_0}, u_{i_0+1}, \ldots, u_t, y)$ are minimal chains, having the same parity, as they are contained in a smaller graph than G. Similarly, $(x, u_1, \ldots, u_{i_0}, v_{j_0})$ and $(x, v_1, \ldots, v_{j_0})$ have the same parity. However, the sum of the length of these four chains, $(s - j_0 + 2) + (t - i_0 + 1) + (i_0 + 1) + j_0 = s + t + 4$, is odd, which is a contradiction.

Theorem 2 (Sachs [13]). *Parity graphs are perfect.*

Without proof, we mention an obvious corollary.

Corollary 3. *A graph $G = (V, E)$ is a parity graph if and only if it does not contain any of the following configurations as induced subgraphs:*
- *Δ_{2k+1} odd cycle, without chord, on $2k+1$ vertices, $k \geq 2$ (also called odd hole).*
- *Δ'_{2k+1} odd cycle, with only one short chord, on $2k+1$ vertices, $k \geq 2$ (a short chord is a chord giving birth to a triangle).*
- *$\tilde{\Delta}_5$ cycle on 5 vertices with two non-crossing chords.*

Here, it seems interesting to recall two related results:

(a) the result of Gallai [5] and Surányi [14] that 'o-triangulated graphs' (in which every odd elementary cycle contains at least two uncrossing chords) are perfect, and

(b) the more general result of Meyniel [11] that each graph which has two chords in every odd elementary cycle is perfect.

In Fig. 4, we give an example of a graph which contains two chords in each of its odd cycles but which is, however, neither an o-triangulated graph nor a parity graph.

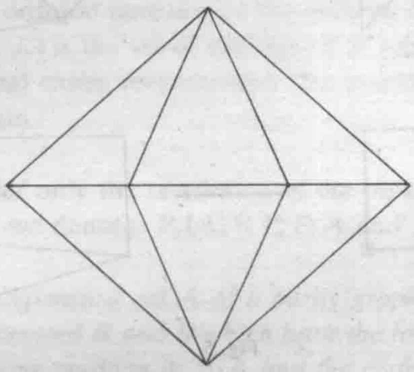

Fig. 4.

3. Description and properties

Notation. For a graph $G=(V,E)$ and $S \subseteq V$ we shall denote by $G(S)$ the subgraph induced by S. We denote by Γ_x the set of vertices adjacent to x, i.e.,

$$\Gamma_x = \{y \in V \mid (x,y) \in E\}.$$

For $A \subseteq V$ the intersection $\Gamma_x \cap A$ will be denoted by $\Gamma_x(A)$. Occasionally, when $H=G(A)$ is a subgraph of G we shall write $\Gamma_x(H)$ for $\Gamma_x(A)$.

Definition 5. We call two vertices x and y *true twins* if they are joined by an edge and have the same adjacents except for x and y (that is, $\Gamma_x \setminus \{y\} = \Gamma_y \setminus \{x\}$). Two vertices x and y are called *false twins* if they are not joined and have the same adjacents.

By Lovász [10] we know that the operation which consists of adding one (true or false) twin to a vertex of a perfect graph builds a new perfect graph. In addition, this operation applied to a parity graph leaves a parity graph. This is false, however for o-triangulated graphs (Fig. 5), but true again for Meyniel graphs [11].

Definition 6. A graph without twins will be called *prime*.

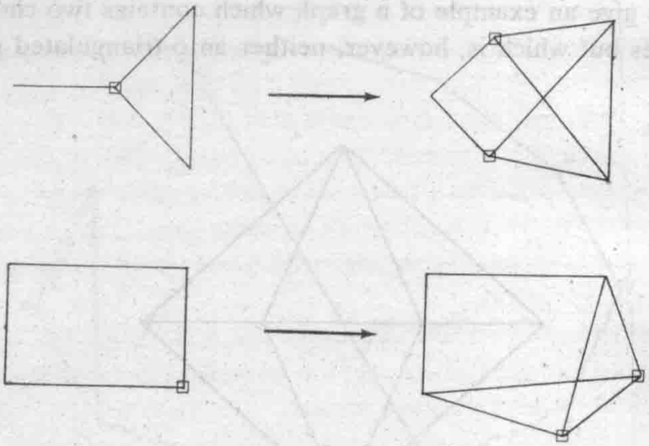

Fig. 5.

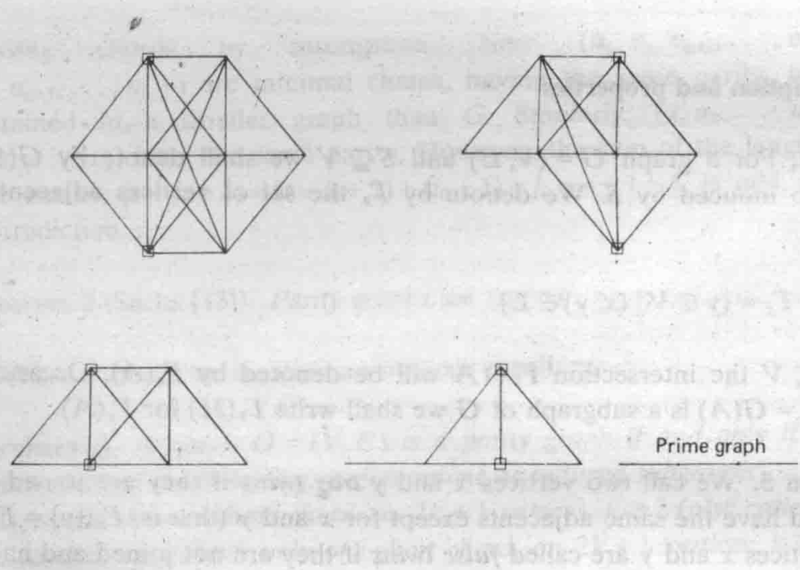

Fig. 6.

In Fig. 6 we give an example of the reduction of a parity graph, and an example of a prime parity graph.

Definition 7. In a parity graph $G = (V, E)$, *the partition induced by the vertex* x, denoted by (P_x, I_x), is the ordered partition of the vertices of G into the classes P_x and I_x, where P_x (resp. I_x) is the set of vertices of V joined to x by an even minimal chain (odd minimal chain, respectively). We assume that x is joined to x by an even minimal chain.

Notation. We may consider only the restriction of the bipartition induced by x to a subset A of V. Then we denote: $P_x(A) = P_x \cap A$ and $I_x(A) = I_x \cap A$.

Lemma 4. *Any minimal separating set A of a parity graph $G = (V, E)$ can be partitioned into two parts denoted R and B which have the following property: the vertices of R induce the same partition in $V \setminus A$ and the vertices of B the opposite partition in $V \setminus A$, that is,*

$$\forall r_1, \forall r_2 \in R \quad P_{r_1}(V\setminus A) = P_{r_2}(V\setminus A) \quad \textit{and hence} \quad I_{r_1}(V\setminus A) = I_{r_2}(V\setminus A),$$

$$\forall r_1 \in R, \forall b_1 \in B \quad P_{r_1}(V\setminus A) = I_{b_1}(V\setminus A) \quad \textit{and hence} \quad I_{r_1}(V\setminus A) = P_{b_1}(V\setminus A).$$

Proof. We suppose $|A| > 1$ (otherwise, it would be obvious). Let CX_1 and CX_2 be two different connected components of the subgraph induced on $V \setminus A$ (cf. Fig. 7).

Let x_1 belong to CX_1, x_2 belong to CX_2, z and t be two different vertices of A. A being minimal there exist minimal chains $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$ joining x_1 and x_2, there only vertex from A being z for $C_1(x_1, x_2)$ and t for $C_2(x_1, x_2)$. Because $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$ have the same parity we have $P_t(V\setminus A) = P_z(V\setminus A)$ or $P_t(V\setminus A) = I_z(V\setminus A)$.

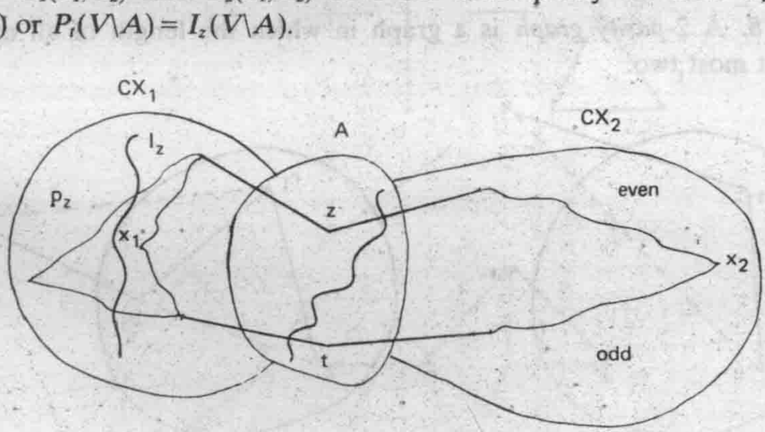

Fig. 7.

Lemma 5. *Let A be a minimal separating set partitioned into R and B as in Lemma 4. If $r_1, r_2 \in R$ and $(r_1, r_2) \in E$, then $\Gamma_{r_1}(V \backslash A) = \Gamma_{r_2}(V \backslash A)$.*

Proof. Suppose the property is false, and let $x \in \Gamma_{r_1}(V \backslash A)$ and $x \notin \Gamma_{r_2}(V \backslash A)$ (cf. Fig. 8). Then (r_2, r_1, x) is a minimal chain of even parity, and hence $x \in I_{r_1}$ and $x \in P_{r_2}$, contradicting Lemma 4.

Lemma 6. *With the same hypotheses as those of the preceding lemma, if $r_1, r_2 \in R$ and $(r_1, r_2) \in E$, then $\Gamma_{r_1}(B) = \Gamma_{r_2}(B)$.*

Proof. Suppose $b_1 \in \Gamma_{r_1}(B)$ and $b_1 \notin \Gamma_{r_2}(B)$, and let x be a vertex in $V \backslash A$ joined to b_1 (such a x exists by minimality of A) (cf. Fig. 9). As (r_2, b_1), (r_1, x) and (r_2, x) are not in E, there is a minimal chain from r_1 to x with length two, and another minimal chain from r_2 to x with length three. This contradicts Lemma 4.

Lemma 7. *With the same hypotheses as before, a connected component of the subgraph induced by the vertices of R has no minimal chain of length three.*

Proof. Suppose there is such a chain (r_1, r_2, r_3, r_4). Let us consider a vertex x of $V \backslash A$ which is adjacent to r_1. From Lemma 5 we know that x is also adjacent to r_2, r_3, and r_4. The subgraph induced by these vertices is a 5-cycle, chords of which may only come from x. This is a contradiction.

Remark 1. This property remains true if for the set R we take the adjacents of any one of the vertices of a parity graph. (They form a separating set which need not be minimal.)

Definition 8. A *2-parity graph* is a graph in which the length of all minimal chains is at most two.

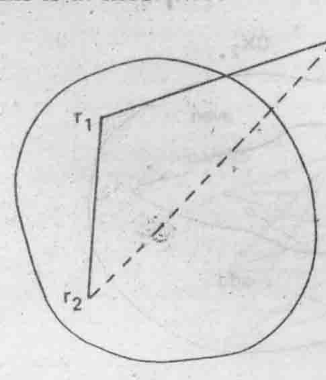

Fig. 8.

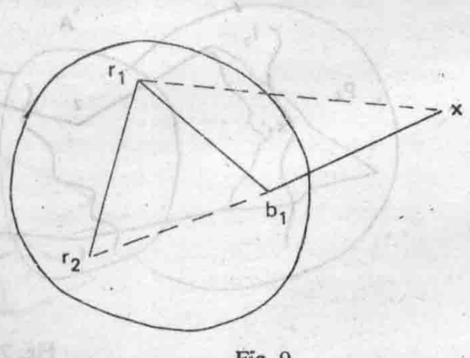

Fig. 9.

Examples of 2-parity graphs are cliques, and, more generally, complete multipartite graphs (a complete multipartite graph is a graph in which the vertices can be partitioned into stable sets, where two vertices are adjacent if they belong to different classes. Such graphs are o-triangulated [5].

Below we give a characteristic property of a 2-parity graph (for other properties, see [3]).

Lemma 8. *A connected* 2*-parity graph with more than one vertex has at least two* (*true, or false*) *twins.*

Proof. For a clique, it is obviously true. Otherwise, let x and y be two vertices which are not joined. There exists a minimal separating set A which separates x from y.

Let CX_x and CX_y be their respective connected components, in the subgraph induced by $V\backslash A$.

For each vertex z of CX_x, z is adjacent to all the vertices of A, otherwise there would be a minimal chain of length three.

When $CX_x = \{x\}$ and $CX_y = \{y\}$, then x and y are false twins, else at least one of CX_x and CX_y has cardinality greater than one, say $|CX_x| > 1$. The proof continues by induction in CX_x. Twins in subgraph CX_x will effectively be twins in the initial graph, because they have the same adjacents in A.

Corollary 9. *A graph is a 2-parity graph if and only if it arises from a single point by adding true or false twins* (*cf. Fig.* 10).

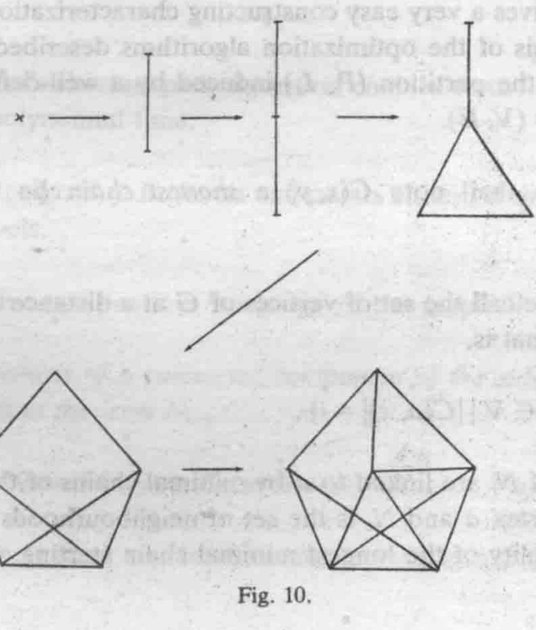

Fig. 10.