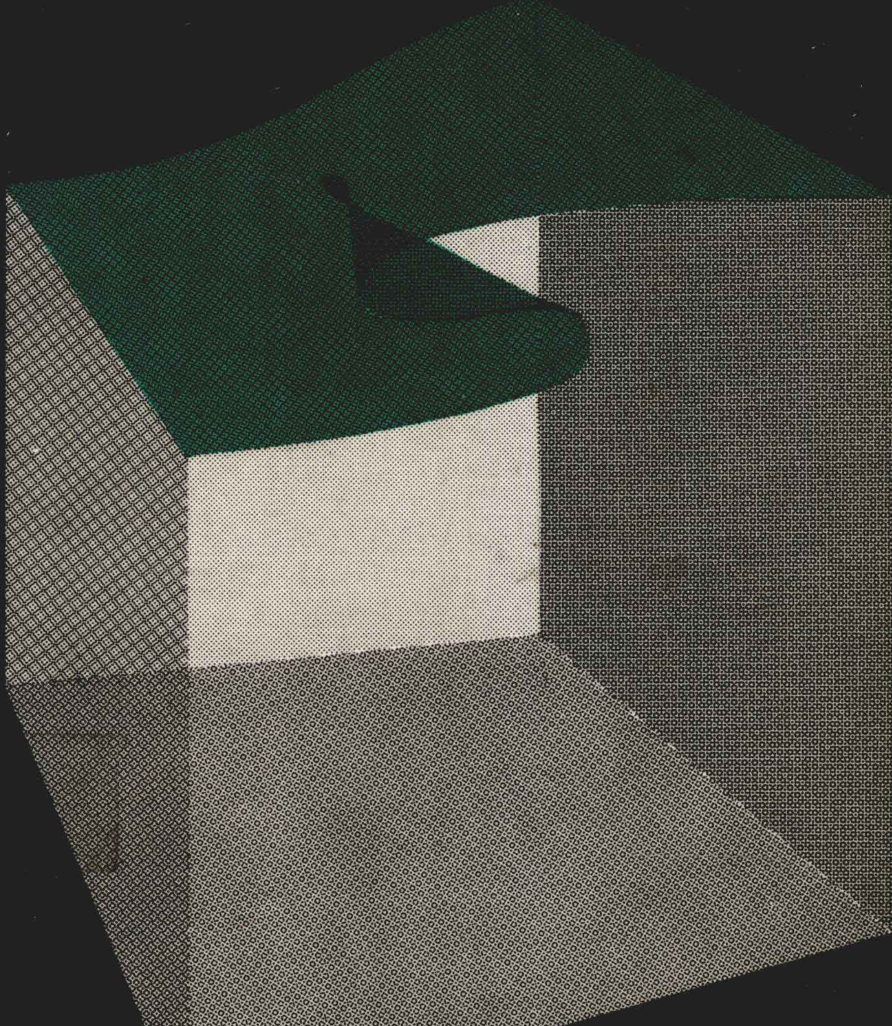
Ellis Horwood Series
MATHEMATICS AND ITS APPLICATIONS

Mathematical Models in the Social, Management and Life Sciences

D. N. Burghes   A. D. Wood



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ELLIS HORWOOD SERIES IN MATHEMATICS AND ITS APPLICATIONS

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ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1980
Reprinted with corrections in 1984 by

ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street, Chichester, West Sussex, PO19 1EB, England

The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:

Australia, New Zealand, South-east Asia:
Jacaranda-Wiley Ltd., Jacaranda Press,
JOHN WILEY & SONS INC.,
G.P.O. Box 859, Brisbane, Queensland 40001, Australia.

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada.

Europe, Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England.

North and South America and the rest of the world:
Halsted Press, a division of
JOHN WILEY & SONS
605 Third Avenue, New York, N.Y. 10016, U.S.A.

British Library Cataloguing in Publication Data

Burghes, David Noel
Mathematical models in the social, management and life sciences. – (Mathematics and its applications).
1. Mathematical models 2. Management – Mathematical models
3. Social Sciences – Mathematical models 4. Biology – Mathematical models
I. Title II. Wood, Alistair D III. Series
001.4'24 QA401 79–40989
ISBN 0–85312–097–8 (Ellis Horwood Ltd., Publishers, Library Edition)
ISBN 0–85312–101–X (Ellis Horwood Ltd., Publishers, Student Edition)
ISBN 0–470–26862–X (Halsted Press)

Typeset in Press Roman by Ellis Horwood Ltd.
Printed in Great Britain by Unwin Brothers Ltd. of Woking

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Preface

In this book we aim to give the reader an appreciation of how mathematical models are formulated, solved and applied, and a concise description of basic mathematical techniques. Each mathematical topic is motivated with a case study; the mathematical technique is presented; the case study solved; and further case studies described. Problems have been set to test the reader's comprehension; and hints and solutions are provided.

The case studies have been taken mainly from management, biology, economics, planning, and sociology: hence our description of the book as 'an account of models in the non-physical sciences' may be seen to be fully justified and earned by its content.

It is written for students in the above disciplines who need a practical course in applying mathematics, and for mathematics students and teachers who want to see the importance of mathematical concepts in a variety of realistic situations.

We have tried to avoid burdening the reader with too many mathematical proofs, at the same time not attempting to conceal any mathematical difficulties, which are fully explained in the text. Our feeling is that over-emphasis of mathematical rigour would detract from our aim of providing an appreciation of the role of mathematics in society today.

In making corrections we wish to thank Mr R. Morrisson of Coldingham, Berwickshire, and various students at N.I.H.E. Dublin for drawing our attention to misprints and errors.

David Burghes
Alastair Wood

CHAPTER 1

Mathematical Modelling: Aims and Philosophy

1.1. MATHEMATICS AND SOCIETY

Few people will deny that the most spectacular successes of mathematics have been in the physical sciences. We have, for instance, the mathematical prediction of the existence of heavenly bodies, subsequently verified by observation, or, in earlier centuries, the formulation of the laws of motion of various bodies. On a more practical level, it could even be claimed that the spread of modern industrial civilisation, for better or for worse, is partly a result of man's ability to solve the differential equations which govern so many of our industrial processes, be they chemical or engineering.

But over the last few decades mathematics has broken out into a whole new range of applications in the social sciences, biology and medicine, management and, it seems, almost every field of human endeavour, providing qualitative, if not quantitative models where none had existed or even been contemplated before. Mathematical techniques now play an important role in planning, managerial decision-making, and economics, which has probably been the longest quantified of the social sciences.

Do we all understand the same thing by 'mathematics'? The man in the street will tend to equate mathematics with arithmetic. But what will children recently exposed to 'modern mathematics' syllabuses in some primary schools equate mathematics with? The engineer will tend to think of the techniques of calculus used to compute solutions to problems. The businessman may think simply of book-keeping. The medical or experimental worker will come up with computers or statistics.

All of these are in part correct, such is the diversity of mathematics, although its unity becomes more obvious with deeper study. Even mathematicians cannot agree. Some, who are usually called applied mathematicians, see mathematics entirely as a model for the physical world. In their view the motivation towards innovation in mathematics arises from the needs of physics. For instance, physicists required a function, with the property that

$$\int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x = f(0)$$

for a wide class of functions. It was found that δ could not exist as an ordinary function, and this led to the development of the theory of generalised functions or distributions.

At the other extreme is the type of pure mathematician who sees mathematics as a formal language constructed from distinct symbols which can be strung together according to well-defined rules to make formulae which have a unique interpretation. Certain formulae are labelled axioms, and others are constructed from them by applying rules of inference. A sequence of such formulae forms a proof. The whole structure of pure mathematics can be set up in this way from a very few axioms, such as 'if X is contained in Y and Y is contained in X, then X equals Y'. By and large the mathematics constructed in this way fits the physical requirements, and occasionally precedes its physical application, for example Kepler's work on the ellipse anticipated an understanding of planetary motion.

Whatever viewpoint we adopt, we must agree on the necessity of having a clear, concise language for transferring thoughts about relatively subtle ideas which may previously have been vague or non-existent. New mathematics is not discovered, but invented: it does not exist until it has been communicated between people. The language must be versatile enough to allow a school-teacher to clarify a point for a pupil while at the same time allowing researchers to be sure that a new result is proved without mistakes.

The contemporary language of mathematics manipulates such basic notions as sets, functions and relations and describes constructions using them. Mathematics is usually laid out according to the following convention.

(i) Definition: this describes a new entity in terms of those that have been defined previously.

(ii) Theorem: this is a statement giving an answer (unfortunately not always complete) to questions raised about these entities. A theorem is sometimes called a Proposition, Corollary, or Lemma.

(iii) Proof: This gives a record of the manipulations necessary to convince the reader that a theorem is a true statement.

Many people regard only these manipulations as mathematics, but the task of formulating the questions and describing the entities is just as important, particularly where modelling is concerned. Ordinary language, for example elementary English, is too imprecise. Suppose that you have never seen a dog and try to find out exactly what it is by using a dictionary. The definition rapidly becomes circular, as we see below:

A 'dog' is an 'animal'
An 'animal' is a 'being'
A 'being' is 'something which exists'
To 'exist' is to 'occur'
To 'occur' is to 'exist'.

There are many paradoxes in the English language. For instance, can you decide whether the statement 'I am lying' is true or false? The advantages of using mathematical language in any situation, but particularly in the new areas mentioned in our introductory paragraphs, may be listed as follows:

(i) The mathematical language is more efficient and less bulky than the written word; it reveals the assumptions being made in their naked simplicity in a way which words do not do efficiently.
(ii) It is more difficult to cheat conclusion with a mathematical argument. The results of a mathematical debate are precise and depend only on the initial assumptions. For a given set of assumptions the mathematical conclusions are accurately expressed, and their results cannot be argued with. It is the assumptions that can and should be criticised.
(iii) With a mathematical description it is possible to arrive at optimal solutions, which would not be obvious without the analysis.

It probably has some disadvantages too. Much time and effort can be spent trying to find solutions for rather irrelevant problems, problems which are so far removed from reality that their solutions have little meaning. In many cases a manager's practical experience and intuition will enable him to make better and quicker decisions than those available to him through a mathematical analysis of the problem. Nevertheless mathematical analysis has had some important successes. For example the technique of linear programming has been extensivly used in transportation problems leading to significant savings in costs; differential equation theory has been used to make precise decisions in glucose tolerance testing for diabetics; and matrix methods are employed in the estimation of future population trends.

Thus we conclude that mathematics has an important role to play in a wide range of applications, so long as we are realistic about what it cannot do, as well as what it can do.

1.2. MATHEMATICAL MODELLING – ITS ROLE AND LIMITATIONS

The underlying theme in all applications of mathematics to real situations is the process of mathematical modelling. By this we mean the problem of translating a real problem from its initial context into a mathematical description, that is, the mathematical model. This mathematical problem is then solved, and the resulting mathematical solutions must be translated back into the original context. The main stages in the modelling process are summarized in Fig. 1.1.

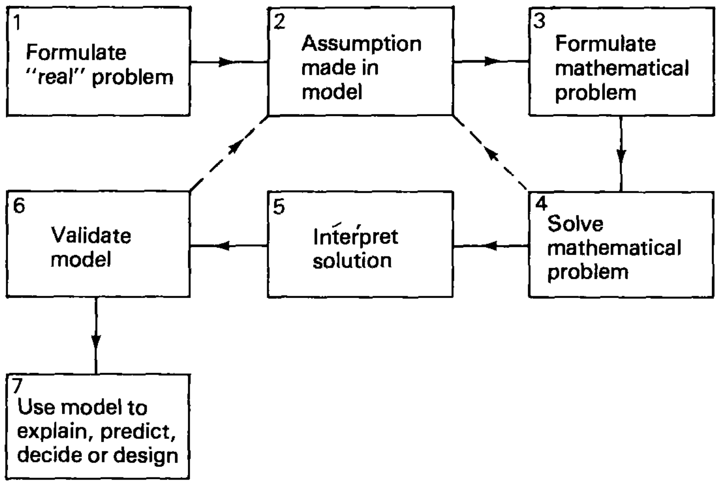

Fig. 1.1.

The left-hand column represents the real world, the right-hand column the mathematical world, and the middle column the connection between these two worlds where firstly the problem is simplified and formalised and secondly the mathematical solution is translated back into the real world situation.

In a straightforward modelling process we move from box 1 to box 7 in sequence, but most modelling is *not* straightforward. We often need to concentrate and spend significantly more time on particular stages. The model is sometimes not adequate for its purpose, and we must move from box 6 back to box 2 and repeat the process, using a more sophisticated model. In many cases, particularly in the social sciences, it is difficult to apply box 6 at all, and we move straight from box 5 to box 7. In other cases it might not be possible to solve the mathematical problem, the mathematics being too complicated to deal with. In this case we return to box 2 and weaken the assumptions. This of course takes us further from the real situation, but leads to an easier mathematical analysis. Whether or not the model is useful will become apparent when we reach box 6 and attempt to validate it.

We can summarise the main stages of modelling into *formulation, solution,* and *application*. The formulation stage is covered by boxes 1, 2 and 3, the solution stage by box 4, and the application stage by boxes 5, 6 and 7. All these stages are important in modelling. We must, though, emphasise that not all modelling will follow this exact pattern. This is just a guide to what modelling is about.

1.3 CASE STUDY

As a simple example to illustrate the modelling process we will consider the problem of a London-based managing director whose company has a factory near the centre of Manchester. Early one morning he is woken by a telephone call from the Manchester factory, where there has been a major industrial dispute during the night shift, and in order to prevent a total walkout of the day staff and a consequent shutdown of the production process (which would be very costly) he is required at the factory in Manchester as soon as possible. What is his problem? To get from his home in the suburbs of London to the factory in Manchester in minimum time. What are his possibilities? To travel by car, train, or plane. So his problem is to choose the form of transport that will get him to his destination as quickly as possible.

We now move from box 1 to box 2 in terms of the modelling diagram in Sec. 1.2 and formulate the model. Each type of journey can be divided into three parts, the time from his home to the starting point of the transport, including the waiting time for the transport, the time on the particular form of transport and the time required from the transport's stopping place to the final destination. To put this into mathematical terms let T_i denote the time required for the ith mode of transport where

$i = 1$ refers to car,
$i = 2$ refers to train,
$i = 3$ refers to plane.

Then

$$\boxed{T_i = a_i + b_i + c_i}$$

where

a_i = time to get from initial point to start of transport i and waiting time;
b_i = time on transport i;
c_i = time from stopping point of transport i to final destination.

In terms of our original modelling diagram we are now moving into box 3. Before we do so it is important to note that we are setting up a general framework which will solve not only this particular managing director's problem but any problems of a similar nature.

On to box 3. Here we state the problem in mathematical terms. We require to find the value of i (1, 2 or 3) which minimises the function T_i, and we can immediately move on to box 4 and write the solution as

$$\boxed{i, \text{ where } T_i = \min (T_1, T_2, T_3)}$$

where the notation

$$\min(x_1, x_2, x_3, \ldots x_n)$$

means choosing the number x_i which is less than all other x values. So our mathematical world is soon dealt with and requires little sophistication. We just need to compare the values T_1, T_2, T_3, and choose the minimum one. The next box is straightforward as well. The interpretation of solution i is that the transport labelled i should be used. There is little we can do to validate the model, and so we move straight on to box 7 and apply the model to the managing director's problem.

The map below illustrates the geographical problem. The managing director lives just outside London at Romford in Essex, and the factory is close to the centre of Manchester. We now estimate each function T_i.

$i = 1$: *CAR*

Clearly for this form of transport $a_i = c_i = 0$, and we just need to evaluate b_i, the actual journey time in the car.

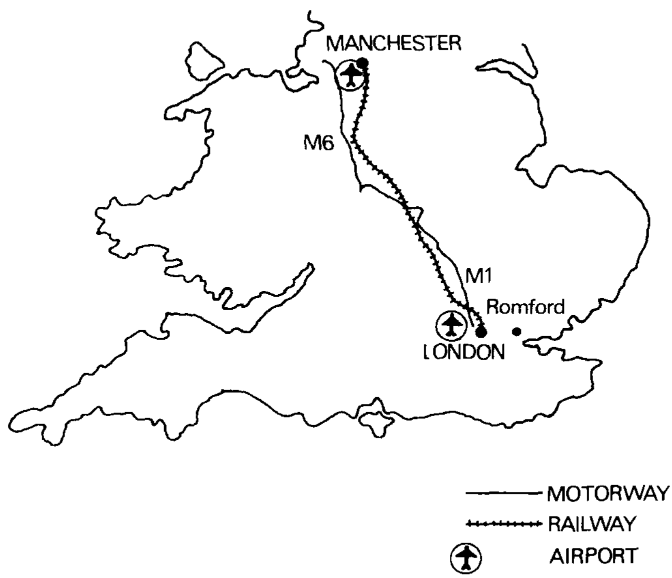

Fig. 1.2

There is first a journey on minor roads across to the M1 which he estimates to take 70 minutes. This is followed by 170 miles of driving at 70 m.p.h. which will take

$$\frac{170}{70} \text{ hours} = 146 \text{ mins.}$$