HEAT CONDUCTION

With Engineering, Geological, and Other Applications

Ingersoll - Zobel - Ingersoll

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PREFACE

The present volume is the successor to and, in effect, a revision of the Ingersoll and Zobel text of some years ago. To quote from the earlier preface: ". . . the theory of heat conduction is of importance, not only intrinsically but also because its broad bearing and the generality of its methods of analysis make it one of the best introductions to more advanced mathematical physics.

"The aim of the authors has been twofold. They have attempted, in the first place, to develop the subject with special reference to the needs of the student who has neither time nor mathematical preparation to pursue the study at great length. To this end, fewer types of problems are handled than in the larger treatises, and less stress has been placed on purely mathematical derivations such as uniqueness, existence, and convergence theorems.

"The second aim has been to point out . . . the many applications of which the results are susceptible . . . . It is hoped that in this respect the subject matter may be of interest to the engineer, for the authors have attempted to select applications with special reference to their technical importance, and in furtherance of this idea have sought and received suggestions from engineers in many lines of work. While many of these applications have doubtless only a small practical bearing and serve chiefly to illustrate the theory, . . . the results in some cases . . . may be found worthy of note. The same may be said of the geological problems.

"While a number of solutions are here presented for the first time . . . no originality can be claimed for the underlying mathematical theory which dates back, of course, to the time of Fourier."

Since the above was written there has been a steady increase in interest in the theory of heat conduction, largely along practical lines. The geologist and geographer are interested in a new tool which will help them in explaining many thermal phenomena and in establishing certain time scales. The engineer, whose use of the theory was formerly limited almost entirely to the steady state, has developed many useful tables and curves for the solution of more general cases and is interested in finding still other methods of attack. The physicist and mathematician have done their part in treating problems which have hitherto resisted solution.

The present volume carries out and extends the aims of the earlier one. Most of the old material has been retained, although revised, and almost an equal amount of new has been added. The geologist, geographer, and engineer will find many new applications discussed, while the mathematician, physicist, and chemist will welcome the addition of a little Bessel function and conjugate function theory, as well as the several extended tables in the appendixes. Some of these are new and have had to be specially evaluated. The number of references has also been greatly enlarged and three-quarters of them are of more recent date than the older volume. A special feature is the extended treatment, particularly as regards applications, of the theory of permanent sources. This is carried out for all three dimensions, but most of the applications center about the two-dimensional case, the most interesting of these being the theory of ground-pipe heat sources for the heat pump. Other features of the revision are a modernized nomenclature, many new problems and illustrations, and the segregation of descriptions of methods of measuring heat-conduction constants in a special chapter.

A feature of particular importance to those whose interests are largely on the practical side is the discussion in Chapter 11 of auxiliary graphical and other approximation methods by which many practical heat conduction problems may be solved with only the simplest mathematics. It is believed that many will appreciate this and in particular the discussion of procedures by which it is possible to handle simply, and with sufficient accuracy for practical purposes, many problems whose solution would be almost impossible by classical methods. As regards the book as a whole, the only mathematical prerequisite necessary for reading it is a reasonable knowledge of calculus. Despite occasional appearances to the contrary, the mathematical theory is not difficult and falls into a pattern which is readily mastered. The authors have tried, in general, to reduce mathematical difficulties to a minimum, and in some cases have deliberately chosen the simpler of two alternate methods of solving a problem, even at a small sacrifice of accuracy.

The authors acknowledge again their indebtedness to the several standard treatises referred to in the preface to the earlier edition, and in particular to Carslaw's "Mathematical Theory of the Conduction of Heat in Solids"; also Carslaw and Jaeger's "Conduction of Heat in Solids." It is hard to single out for special credit any of the hundred-odd other books and papers to which they are indebted and which are listed at the end of this volume, but perhaps particular reference should be made to McAdams' "Heat Transmission" and to papers by Emmons, Newman, and Olson and Schultz.

The authors are glad to acknowledge assistance from many friends. These include: O. A. Hougen, D. W. Nelson, F. E. Volk, and M. O. Withey of the College of Engineering, University of Wisconsin; J. D. MacLean of the Forest Products Laboratory; J. H. Van Vleck of Harvard University, W. J. Mead of Massachusetts Institute of Technology, and A. C. Lane of Cambridge; C. E. Van Orstrand, formerly of the U.S. Geological Survey; H. W. Norton of Oak Ridge, Tennessee; C. C. Furnas of the Curtiss-Wright Corp., B. Kelley of the Bell Aircraft Corp., and G. H. Zenner and L. D. Potts of the Linde Air Products Laboratory, in Buffalo; A. C. Crandall of the Indianapolis Light and Power Co.; M. S. Oldacre of the Utilities Research Commission in Chicago; and a large number of others who have given help and suggestions. The authors are particularly indebted to F. T. Adler of the Department of Physics of the University of Wisconsin and to H. W. March of the Department of Mathematics for much assistance; also to K. J. Arnold of the same department and to Mrs. M. H. Glissendorf and Miss R. C. Bernstein of the university computing service

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THE AUTHORS

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PREFACE TO THE REVISED EDITION

In the present volume, which is really the third edition of this book, the principal changes, relative to the last or McGraw-Hill edition, are the addition of Chapters 13 and 14.

The first of these has to do with the heat pump. No excuse is needed for adding to the theory of this modern system of year-round air-conditioning, which is today receiving so much attention. The use of the ground as a source of heat for the heat pump involves a number of phases of heat conduction theory and is one of the best possible applications of such theory. The simple introductory treatment given in the last edition has here been greatly expanded.

In the last chapter two of the newer applications of the heat flow equation are discussed, *viz.*, the subjects of drying and of soil consolidation—both fields of increasing importance.

The new material adds almost 20 per cent to the size of the book, 25 per cent to the number of figures, and it is hoped an even larger proportional value to the volume as a whole.

We are glad to acknowledge assistance from many friends in preparing this edition and we are especially indebted to Professor W. R. Marshall, Jr., of the Department of Chemical Engineering and Professor D. W. Nelson of the Department of Mechanical Engineering of the University of Wisconsin, and to Professor Jack E. McKee of the Department of Civil Engineering of California Institute of Technology.

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CHAPTER 1

INTRODUCTION

1.1. Symbols. The following table lists the principal symbols and abbreviations used in this book. They have been chosen in agreement, so far as practicable, with the recommendations of the American Standards Association and with general scientific practice.

TABLE 1.1.—NOMENCLATURE

A	Area, cm^2 or ft^2.
α	Thermal diffusivity, cgs or fph (Secs. 1.3, 1.5, Appendix A).
$B(x)$	$2(e^{-x} - e^{-4x} + e^{-9x} - \cdots)$ (Sec. 9.17, Appendix H).
$B_a(x)$	$\frac{6}{\pi^2}\left(e^{-x} + \frac{1}{4}e^{-4x} + \frac{1}{9}e^{-9x} + \cdots\right)$ (Sec. 9.18, Appendix H).
β, γ	Variables of integration; also constants.
λ	Variable of integration; also a constant; also wave length.
Btu	British thermal unit, 1 lb water 1°F (Sec. 1.5).
c	Specific heat (constant pressure), cal/(gm)(°C), or Btu/(lb)(°F); also a constant.
cal	Calorie, 1 gm water 1°C (Sec. 1.5).
cgs	Centimeter-gram-second system; used here only with centigrade temperature scale and calorie as unit of heat.
$C(x)$	$2\left(\frac{e^{-xz_1^2}}{z_1J_1(z_1)} + \frac{e^{-xz_2^2}}{z_2J_1(z_2)} + \frac{e^{-xz_3^2}}{z_3J_1(z_3)} + \cdots\right)$ (Sec. 9.38, Appendix J).
$\exp x$	e^x.
fph	Foot-pound-hour system, used here only with Fahrenheit temperature scale and Btu as heat unit.
h	Coefficient of heat transfer between a surface and its surroundings, cal/(sec)(cm^2)(°C) or Btu/(hr)(ft^2)(°F); sometimes called "emissivity" or "exterior conductivity" (Sec. 2.5, Appendix A).
η	$\frac{1}{2\sqrt{\alpha t}}$.
$I(x)$	$\int_x^\infty \beta^{-1}e^{-\beta^2}\,d\beta$ (Sec. 9.8, Appendix F).
$J_n(x)$	Bessel function (Sec. 9.36).
k	Thermal conductivity, cgs or fph (Secs. 1.3, 1.5, Appendix A).
$\ln x$	$\log_e x$.

TABLE 1.1.—NOMENCLATURE—(*Continued*)

$\Phi(x)$	Probability integral, $\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}\,d\beta$ (Appendix D).
Q	Quantity of heat, cal or Btu (sometimes taken per unit length or unit area; see Q').
q	Rate of heat flow, cal/sec or Btu/hr (sometimes also used for rate of heat production).
Q'	Rate of heat production or withdrawal in permanent sources or sinks, cal/sec or Btu/hr for three-dimensional case; cal/sec per cm length or Btu/hr per ft length for two-dimensional case; cal/(sec)(cm^2) or Btu/(hr)(ft^2) for one-dimensional case (Secs. 8.2, 9.9).
ρ	Density, gm/cm^3, or lb/ft^3.
R	Thermal resistance $\frac{x}{kA}$ (Sec. 3.3).
S	Strength of instantaneous source, $\frac{Q}{c\rho}$ (Secs. 8.2, 9.9).
S'	Strength of permanent source, $\frac{Q'}{c\rho}$ (Secs. 8.2, 9.9).
$S(x)$	$\frac{4}{\pi}\left(e^{-\pi^2 x} - \frac{1}{3}e^{-9\pi^2 x} + \frac{1}{5}e^{-25\pi^2 x} - \cdots\right)$ (Sec. 8.16, Appendix G).
t	Time, seconds or hours.
T*	Temperature, °C or °F.
w	Rate of flow of heat per unit area, $\frac{q}{A}$; cal/(sec)(cm^2) or Btu/(hr)(ft^2) (Sec. 1.3)

1.2. Historical. The mathematical theory of heat conduction in solids, the subject of principal concern in this book, is due principally to Jean Baptiste Joseph Fourier (1768–1830) and was set forth by him in his "Théorie analytique de la chaleur."[42]† While Lambert, Biot, and others had developed some more or less correct ideas on the subject, it was Fourier who first brought order out of the confusion in which the experimental physicists had left the subject. While Fourier treated a large number of cases, his work was extended and applied to more complicated problems by his contemporaries Laplace and Poisson, and later by a number of others, including Lamé, Sir W. Thomson[146,147] (Lord Kelvin), and Riemann.[160] To the

* The use of θ for temperature, as in the former edition of this book, has been discontinued here, partly because many modern writers attach the significance of time to it and partly because of the increasing adoption of T. It is suggested that, to avoid confusion, this be always pronounced "captee."

† Superscript figures throughout the text denote references in Appendix M.

last mentioned writer all students of the subject should feel indebted for the very readable form in which he has put much of Fourier's work. The most authoritative recent work on the subject is that of Carslaw and Jaeger.[27a]

1.3. Definitions. When different parts of a solid body are at different temperatures, heat flows from the hotter to the colder portions by a process of electronic and atomic energy transfer known as "conduction." The rate at which heat will be transferred has been found by experiment to depend on a number of conditions that we shall now consider.

To help visualize these ideas imagine in a body two parallel planes or laminae of area A and distance x apart, over each of which the temperature is constant, being T_1 in one case and T_2 in the other. Heat will then flow from the hotter of these isothermal surfaces to the colder, and the quantity Q that will be conducted in time t will be given by

$$Q = k\,\frac{T_1 - T_2}{x}\,At \qquad (a)$$

or

$$q = \frac{dQ}{dt} = k\,\frac{T_1 - T_2}{x}\,A \qquad (b)$$

where k is a constant for any given material known as the *thermal conductivity* of the substance. It is then numerically equal to the quantity of heat that flows in unit time through unit area of a plate of unit thickness having unit temperature difference between its faces.

The limiting value of $(T_2 - T_1)/x$ or $\partial T/\partial x$ is known as the *temperature gradient* at any point. If due attention is paid to sign, we see that if $\partial T/\partial x$ is taken in the direction of heat flow it is intrinsically negative. Hence, if we wish to have a positive value for the rate at which heat is transferred across an isothermal surface in a positive direction, we write

$$q = -kA\,\frac{\partial T}{\partial x} \qquad (c)$$

or

$$w = -k\,\frac{\partial T}{\partial x} \qquad (d)$$

where $w\ (= q/A)$ is called the "flux" of heat across the surface at that point. If instead of an isothermal surface we consider another, making an angle ϕ with it, we can see that both the flux across the surface and the temperature gradient across the normal to such surface will be diminished, the factor being cos ϕ, so that we may write in general for the flux across any surface

$$w = -k \frac{\partial T}{\partial n} \tag{e}$$

where the derivative is taken along the outward drawn normal, *i.e.*, in the direction of decreasing temperature. This shows that the direction of (maximum) heat flow is normal to the isotherms.

While the rate at which heat is transferred in a body, *e.g.*, along a thermally insulated rod, is dependent only on the conductivity and other factors noted, the rise in temperature that this heat will produce will vary with the specific heat c and the density ρ of the body. We must then introduce another constant α whose significance will be considered later, determined by the relation

$$\alpha = \frac{k}{c\rho} \tag{f}$$

The constant α has been termed by Kelvin the *thermal diffusivity* of the substance, and by Maxwell its *thermometric conductivity*.

Equations (*a*) and (*e*) express what is sometimes referred to as the fundamental hypothesis of heat conduction. Its justification or proof rests on the agreement of calculations made on this hypothesis, with the results of experiment, not only for the very simple but for the more complicated cases as well.

1.4. Fields of Application. From (1.3*a*) we may infer in what field the results of our study will find application. We may conclude first that our derivations will hold good for any body in which heat transfer takes place according to this law, if k is the same for all parts and all directions in the body. This includes all homogeneous isotropic solids and also liquids and gases in cases where convection and radiation are negligible. The equation also shows that, since only differences of temperature are involved, the actual temperature of the system is immaterial. We shall have cause to remember this statement frequently; for, while many cases are derived on the supposition that the temperature at the boundary is zero, the results are made applicable to cases in which this is any other constant temperature by a simple shift of the temperature scale.

But the results of the study of heat conduction are not limited in their application to heat alone, for parts of the theory find application in certain gravitational problems, in static and current electricity, and in elasticity, while the methods developed are of very general application in mathematical physics. As an example of such relationship to other fields it may be pointed out that, if T in (1.3a) is interpreted as electric potential and k as electric conductivity, we have the law of the flow of electricity and all our derivations may be interpreted accordingly.

Another field of application is in drying of porous solids, *e.g.*, wood. It is found that for certain stages of drying the moisture flow is fairly well represented* by the heat-conduction equation. In this case Q represents the amount of water (or other liquid) transferred by diffusion, T is the moisture content in unit volume of the (dry) solid, k is the rate of moisture flow per unit area for unit concentration gradient. The quantity $c\rho$, which normally represents the amount of heat required to raise the temperature of unit volume of the substance by one degree, is here the amount of water required to raise the moisture content of unit volume by unit amount. This is obviously unity, so k and α are the same in this case; k is here called the "diffusion constant." The passage of liquid through a porous solid, as in drying, is a more complicated process than heat flow, and the application of conduction theory has definite limitations, as pointed out by Hougen, McCauley, and Marshall.[58] It may be added that in all probability the diffusion of gas in a metal is subject to the same general theory as water diffusion in porous materials.

Lastly, we may mention the work of Biot[15] on settlement and consolidation of soils. This indicates that the conduction

* Bateman, Hohf and Stamm,[8] Ceaglske and Hougen,[29] Gilliland and Sherwood,[45] Lewis,[75] McCready and McCabe,[91] Newman,[101] Sherwood,[127,128] and Tuttle.[150]

equation may play an important part in the theory of these phenomena.

1.5. Units; Dimensions. Two consistent systems of conductivity units are in common use, having as units of length, mass, time, and temperature, respectively, the centimeter, gram, second, and centigrade degree, on the one hand and the foot, pound, hour, and Fahrenheit degree on the other. The former unit will be referred to as cgs and the latter as fph as regards system. This gives as the unit of heat in the first case the (small) calorie, or heat required to raise the temperature of 1 gm of water 1°C, frequently specified at 15°C; and in the second the Btu, or heat required to raise 1 lb of water 1°F, sometimes specified at 39.1°F* and sometimes at 60°F. The cgs thermal-conductivity unit is the calorie per second, per square centimeter of area, for a temperature gradient of 1°C per centimeter, which shortens to cal/(sec)(cm)(°C), while the fph conductivity unit is the Btu/(hr)(ft)(°F). Similarly, the units of diffusivity come out cm^2/sec and ft^2/hr. The unit in frequent use in some branches of engineering having areas in square feet but temperature gradients expressed in degrees per *inch* will not be used here because of difficulties attendant on the use of two different units of length.

In converting thermal constants from one system to another and in solving many problems Table 1.2 will be found useful.

Conversion factors other than those listed above may be readily derived from a consideration of the dimensions of the units. From (1.3*a*)

$$k = \frac{Q}{T_1 - T_2}\frac{x}{At} \qquad (a)$$

Since—putting the matter as simply as possible—the unit of heat is that necessary to raise unit mass of water one degree, its dimensions are mass and temperature; thus, the dimensions of $Q/(T_1 - T_2)$ are simply M. Hence, K the unit of conductivity is the unit of mass M divided by the units of length L

* The matter of whether heat units are specified for the temperature of maximum density of water or for a slightly higher temperature may result in discrepancies of the order of half a percent, but this is of little practical importance since this is below the usual limit of error in thermal conductivity work.