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Philippe Biane
Luc Bouten
Fabio Cipriani
Norio Konno
Nicolas Privault
Quanhua Xu

Quantum Potential Theory

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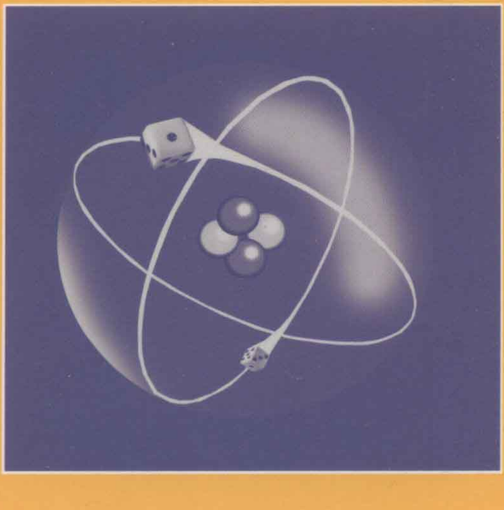
Michael Schürmann



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Preface

This volume contains the notes of lectures given at the School "Quantum Potential Theory: Structure and Applications to Physics". This school was held at the Alfried Krupp Wissenschaftskolleg in Greifswald from February 26 to March 9, 2007. We thank the lecturers for the hard work they accomplished in preparing and giving these lectures and in writing these notes. Their lectures give an introduction to current research in their domains, which is essentially self-contained and should be accessible to Ph.D. students. We hope that this volume will help to bring together researchers from the areas of classical and quantum probability, functional analysis and operator algebras, and theoretical and mathematical physics, and contribute in this way to developing further the subject of quantum potential theory.

We are greatly indebted to the Alfried Krupp von Bohlen und Halbach-Stiftung for the financial support, without which the school would not have been possible. We are also very thankful for the support by the University of Greifswald and the University of Franche-Comté. One of the organisers (UF) was supported by a Marie Curie Outgoing International Fellowship of the EU (Contract Q-MALL MOIF-CT-2006-022137).

Special thanks go to Melanie Hinz who helped with the preparation and organisation of the school and who took care of all of the logistics.

Finally, we would like to thank all the students for coming to Greifswald and helping to make the school a success.

Sendai and Greifswald,
June 2008

Uwe Franz
Michael Schürmann

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Introduction

The term potential theory comes from 19th century physics, where the fundamental forces like gravity or electrostatic forces were described as the gradients of potentials, i.e. functions which satisfy the Laplace equation. Hence potential theory was the study of solutions of the Laplace equation. Nowadays the fundamental forces in physics are described by systems of non-linear partial differential equations such as the Einstein equations and the Yang-Mills equations, and the Laplace equation arises only as a limiting case. Nevertheless, the Laplace equation is still used in applications in many areas of physics and engineering like heat conduction and electrostatics. And the term "potential theory" has survived as a convenient label for the theory of functions satisfying the Laplace equation, i.e. so-called harmonic functions.

In the 20th century, with the development of probability and stochastic processes, it was discovered that potential theory is intimately related to the theory of Markov processes, in particular diffusion processes and Brownian motion. The distributions of these processes evolve according to a heat equation, and invariant distributions satisfy a Laplace-type equation. Conversely, these processes can be used to express solutions of, e.g., the Laplace equation. For more details see Nicolas Privault's lecture "Potential Theory in Classical Probability" in this volume.

The notions of quantum stochastic processes and quantum Markov processes were introduced in the 1970's and allow to describe open quantum systems in close analogy to classical probability and classical Markov processes. Roughly speaking, one can now recognize two different trends in the subsequent development of the theory of quantum Markov processes. The first is guided by physical applications, studies concrete physically motivated models, and develops tools for filtering noisy quantum signals or controling noisy quantum systems. The second aims to develop a mathematical theory, by generalizing or extending key results of the theory of Markov processes to the quantum (or noncommutative) case, and by looking for analogues of important tools that greatly influenced the development of classical potential theory, like stochastic calculus, Dirichlet forms, or boundaries of random

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walks. In our school the first direction was represented by Luc Bouten's lecture "Applications of Controlled Quantum Processes in Quantum Optics", the second by Philippe Biane's lecture "Introduction to Random Walks on Noncommutative Spaces" and by Fabio Cipriani's lecture "Noncommutative Dirichlet Forms", see also the corresponding chapters of this book.

Besides providing important background material on operator algebras and noncommutative analogues of function spaces used in other lectures, Quanhua Xu's lecture on "Interactions between Quantum Probability and Operator Space Theory" shows how quantum probability can be applied to modern functional analysis. For example, a clever choice of sequences of quantum random variables plays an essential role in establishing key results like noncommutative Khintchine type inequalities.

Central questions from probabilistic potential theory like the computation of hitting times and the study of the asymptotic behaviour of a walk are also the main topic in Norio Konno's lecture on "Quantum Walks". These quantum walks are not quantum Markov processes in the sense of the lectures by Biane, Bouten, and Cipriani, but another type of quantum analogue of random walks and Markov chains, and many of the classical potential theoretical methods have interesting analogues adapted to this case. By giving an introduction and survey of this quickly developing field this lecture was an enrichment of the school and nicely complements the other chapters.

The goal of the School "Quantum Potential Theory: Structure and Applications to Physics" and these lecture notes is two-fold. First of all we want to provide an introduction to the rapidly developing theory of quantum Markov semigroups and quantum Markov processes with its manifold aspects ranging from functional analysis and probability theory to quantum physics. We hope that we have succeeded in preparing a monograph that is accessible to graduate students in mathematics and physics. But furthermore we also hope that this book will catch the interest of experienced mathematicians and physicists working in this field or related fields, in order to stimulate more communication between researchers working on "pure" and "applied" aspects. We believe that a strong collaboration between these communities will be to everybody's benefit. Keeping in mind the physical applications will help to sharpen the theoreticians' eye for the relevant questions and properties, and new powerful mathematical tools will allow to get a better and deeper understanding of concrete physical systems.

Potential Theory in Classical Probability

Nicolas Privault

Abstract These notes are an elementary introduction to classical potential theory and to its connection with probabilistic tools such as stochastic calculus and the Markov property. In particular we review the probabilistic interpretations of harmonicity, of the Dirichlet problem, and of the Poisson equation using Brownian motion and stochastic calculus.

1 Introduction

The origins of potential theory can be traced to the physical problem of reconstructing a repartition of electric charges inside a planar or a spatial domain, given the measurement of the electrical field created on the boundary of this domain.

In mathematical analytic terms this amounts to representing the values of a function h inside a domain given the data of the values of h on the boundary of the domain. In the simplest case of a domain empty of electric charges, the problem can be formulated as that of finding a harmonic function h on E (roughly speaking, a function with vanishing Laplacian, see § 2.2 below), given its values prescribed by a function f on the boundary ∂E, i.e. as the Dirichlet problem:

$$\begin{cases} \Delta h(y) = 0, & y \in E, \\ \\ h(y) = f(y), & y \in \partial E. \end{cases}$$

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Close connections between the notion of potential and the Markov property have been observed at early stages of the development of the theory, see e.g. [Doo84] and references therein. Thus a number of potential theoretic problems have a probabilistic interpretation or can be solved by probabilistic methods.

These notes aim at gathering both analytic and probabilistic aspects of potential theory into a single document. We partly follow the point of view of Chung [Chu95] with complements on analytic potential theory coming from Helms [Hel69], some additions on stochastic calculus, and probabilistic applications found in Bass [Bas98].

More precisely, we proceed as follow. In Section 2 we give a summary of classical analytic potential theory: Green kernels, Laplace and Poisson equations in particular, following mainly Brelot [Bre65], Chung [Chu95] and Helms [Hel69]. Section 3 introduces the Markovian setting of semigroups which will be the main framework for probabilistic interpretations. A sample of references used in this domain is Ethier and Kurtz [EK86], Kallenberg [Kal02], and also Chung [Chu95]. The probabilistic interpretation of potential theory also makes significant use of Brownian motion and stochastic calculus. They are summarized in Section 4, see Protter [Pro05], Ikeda and Watanabe [IW89], however our presentation of stochastic calculus is given in the framework of normal martingales due to their links with quantum stochastic calculus, cf. Biane [Bia93]. In Section 5 we present the probabilistic connection between potential theory and Markov processes, following Bass [Bas98], Dynkin [Dyn65], Kallenberg [Kal02], and Port and Stone [PS78]. Our description of the Martin boundary in discrete time follows that of Revuz [Rev75].

2 Analytic Potential Theory

2.1 Electrostatic Interpretation

Let E denote a closed region of $\mathbb{R}^n$, more precisely a compact subset having a smooth boundary ∂E with surface measure σ. Gauss's law is the main tool for determining a repartition of electric charges inside E, given the values of the electrical field created on ∂E. It states that given a repartition of charges $q(dx)$ the flux of the electric field $\mathbf{U}$ across the boundary ∂E is proportional to the sum of electric charges enclosed in E. Namely we have

$$\int_E q(dx) = \epsilon_0 \int_{\partial E} \langle \mathbf{n}(x), \mathbf{U}(x) \rangle \sigma(dx), \tag{2.1}$$

where $q(dx)$ is a signed measure representing the distribution of electric charges, $\epsilon_0 > 0$ is the electrical permittivity constant, $\mathbf{U}(x)$ denotes the electric field at $x \in \partial E$, and $\mathbf{n}(x)$ represents the outer (i.e. oriented towards the exterior of E) unit vector orthogonal to the surface ∂E.

On the other hand the divergence theorem, which can be viewed as a particular case of the Stokes theorem, states that if $\mathbf{U} : E \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field we have

$$\int_E \operatorname{div} \mathbf{U}(x) dx = \int_{\partial E} \langle \mathbf{n}(x), \mathbf{U}(x) \rangle \sigma(dx), \tag{2.2}$$

where the divergence $\operatorname{div} \mathbf{U}$ is defined as

$$\operatorname{div} \mathbf{U}(x) = \sum_{i=1}^{n} \frac{\partial \mathbf{U}_i}{\partial x_i}(x).$$

The divergence theorem (2.2) can be interpreted as a mathematical formulation of the Gauss law (2.1). Under this identification, $\operatorname{div} \mathbf{U}(x)$ is proportional to the density of charges inside E, which leads to the Maxwell equation

$$\epsilon_0 \operatorname{div} \mathbf{U}(x) dx = q(dx), \tag{2.3}$$

where $q(dx)$ is the distribution of electric charge at x and $\mathbf{U}$ is viewed as the induced electric field on the surface ∂E.

When $q(dx)$ has the density $q(x)$ at x, i.e. $q(dx) = q(x)dx$, and the field $\mathbf{U}(x)$ derives from a potential $V : \mathbb{R}^n \to \mathbb{R}_+$, i.e. when

$$\mathbf{U}(x) = \nabla V(x), \qquad x \in E,$$

Maxwell's equation (2.3) takes the form of the Poisson equation:

$$\epsilon_0 \Delta V(x) = q(x), \qquad x \in E, \tag{2.4}$$

where the Laplacian $\Delta = \operatorname{div} \nabla$ is given by

$$\Delta V(x) = \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i^2}(x), \qquad x \in E.$$

In particular, when the domain E is empty of electric charges, the potential V satisfies the Laplace equation

$$\Delta V(x) = 0 \qquad x \in E.$$

As mentioned in the introduction, a typical problem in classical potential theory is to recover the values of the potential $V(x)$ inside E from its values on the boundary ∂E, given that $V(x)$ satisfies the Poisson equation (2.4). This can be achieved in particular by representing $V(x)$, $x \in E$, as an integral with respect to the surface measure over the boundary ∂E, or by direct solution of the Poisson equation for $V(x)$.

Consider for example the Newton potential kernel

$$V(x) = \frac{q}{\epsilon_0 s_n} \frac{1}{\|x-y\|^{n-2}}, \qquad x \in \mathbb{R}^n \backslash \{y\},$$

created by a single charge q at $y \in \mathbb{R}^n$, where $s_2 = 2\pi$, $s_3 = 4\pi$, and in general

$$s_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad n \geq 2,$$

is the surface of the unit $n-1$-dimensional sphere in $\mathbb{R}^n$.

The electrical field created by V is

$$\mathbf{U}(x) := \nabla V(x) = \frac{q}{\epsilon_0 s_n} \frac{x-y}{\|x-y\|^{n-1}}, \qquad x \in \mathbb{R}^n \backslash \{y\},$$

cf, Figure 1. Letting $B(y,r)$, resp. $S(y,r)$, denote the open ball, resp. the sphere, of center $y \in \mathbb{R}^n$ and radius $r > 0$, we have

$$\begin{aligned}\int_{B(y,r)} \Delta V(x) dx &= \int_{S(y,r)} \langle \mathbf{n}(x), \nabla V(x) \rangle \sigma(dx) \\ &= \int_{S(y,r)} \langle \mathbf{n}(x), \mathbf{U}(x) \rangle \sigma(dx) \\ &= \frac{q}{\epsilon_0},\end{aligned}$$

where σ denotes the surface measure on $S(y,r)$.
From this and the Poisson equation (2.4) we deduce that the repartition of electric charge is

$$q(dx) = q\delta_y(dx)$$

i.e. we recover the fact that the potential V is generated by a single charge located at y. We also obtain a version of the Poisson equation (2.4) in distribution sense:

$$\Delta_x \frac{1}{\|x-y\|^{n-2}} = s_n \delta_y(dx),$$

where the Laplacian Δ_x is taken with respect to the variable x. On the other hand, taking $E = B(0,r) \backslash B(0,\rho)$ we have $\partial E = S(0,r) \cup S(0,\rho)$ and

$$\begin{aligned}\int_E \Delta V(x) dx &= \int_{S(0,r)} \langle \mathbf{n}(x), \nabla V(x) \rangle \sigma(dx) + \int_{S(0,\rho)} \langle \mathbf{n}(x), \nabla V(x) \rangle \sigma(dx) \\ &= cs_n - cs_n = 0,\end{aligned}$$

hence

$$\Delta_x \frac{1}{\|x-y\|^{n-2}} = 0, \qquad x \in \mathbb{R}^n \backslash \{y\}.$$

The electrical permittivity ϵ_0 will be set equal to 1 in the sequel.

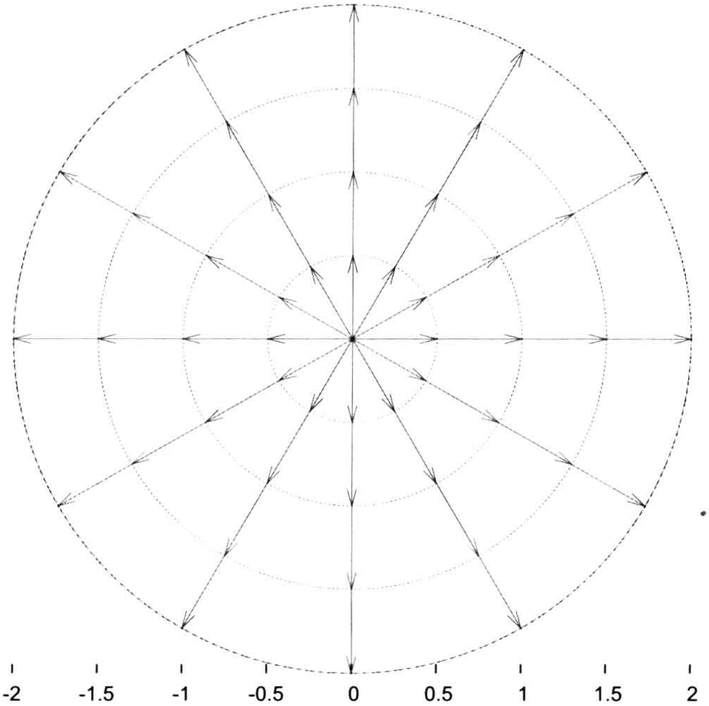

Fig. 1 Electrical field generated by a point mass at $y = 0$.

2.2 Harmonic Functions

The notion of harmonic function will be first introduced from the mean value property. Let

$$\sigma_r^x(dy) = \frac{1}{s_n r^{n-1}} \sigma(dy)$$

denote the normalized surface measure on $S(x, r)$, and recall that

$$\int f(x)dx = s_n \int_0^\infty r^{n-1} \int_{S(y,r)} f(z)\sigma_r^y(dz)dr.$$

Definition 2.1. A continuous real-valued function on an open subset O of $\mathbb{R}^n$ is said to be *harmonic*, resp. superharmonic, in O if one has

$$f(x) = \int_{S(x,r)} f(y)\sigma_r^x(dy),$$

resp.

$$f(x) \geq \int_{S(x,r)} f(y)\sigma_r^x(dy),$$

for all $x \in O$ and $r > 0$ such that $B(x, r) \subset O$.

Next we show the equivalence between the mean value property and the vanishing of the Laplacian.

Proposition 2.2. *A $\mathcal{C}^2$ function f is harmonic, resp. superharmonic, on an open subset O of $\mathbb{R}^n$ if and only if it satisfies the Laplace equation*

$$\Delta f(x) = 0, \qquad x \in O,$$

resp. the partial differential inequality

$$\Delta f(x) \leq 0, \qquad x \in O.$$

Proof. In spherical coordinates, using the divergence formula and the identity

$$\begin{aligned} \frac{d}{dr}\int_{S(0,1)} f(y+rx)\sigma_1^0(dx) &= \int_{S(0,1)} \langle x, \nabla f(y+rx)\rangle \sigma_1^0(dx) \\ &= \frac{r}{s_n}\int_{B(0,1)} \Delta f(y+rx)dx, \end{aligned}$$

yields

$$\begin{aligned} \int_{B(y,r)} \Delta f(x)dx &= r^{n-1}\int_{B(0,1)} \Delta f(y+rx)dx \\ &= s_n r^{n-2}\int_{S(0,1)} \langle x, \nabla f(y+rx)\rangle \sigma_1^0(dx) \\ &= s_n r^{n-2}\frac{d}{dr}\int_{S(0,1)} f(y+rx)\sigma_1^0(dx) \\ &= s_n r^{n-2}\frac{d}{dr}\int_{S(y,r)} f(x)\sigma_r^y(dx). \end{aligned}$$

If f is harmonic, this shows that

$$\int_{B(y,r)} \Delta f(x)dx = 0,$$

for all $y \in E$ and $r > 0$ such that $B(y,r) \subset O$, hence $\Delta f = 0$ on O. Conversely, if $\Delta f = 0$ on O then

$$\int_{S(y,r)} f(x)\sigma_r^y(dx)$$

is constant in r, hence

$$f(y) = \lim_{\rho\to 0}\int_{S(y,\rho)} f(x)\sigma_\rho^y(dx) = \int_{S(y,r)} f(x)\sigma_r^y(dx), \qquad r > 0.$$

The proof is similar in the case of superharmonic functions. □

The fundamental harmonic functions based at $y \in \mathbb{R}^n$ are the functions which are harmonic on $\mathbb{R}^n \setminus \{y\}$ and depend only on $r = \|x-y\|$, $y \in \mathbb{R}^n$. They satisfy the Laplace equation

$$\Delta h(x) = 0, \qquad x \in \mathbb{R}^n,$$

in spherical coordinates, with

$$\Delta h(r) = \frac{d^2 h}{dr^2}(r) + \frac{(n-1)}{r}\frac{dh}{dr}(r).$$

In case $n = 2$, the fundamental harmonic functions are given by the logarithmic potential

$$h_y(x) = \begin{cases} -\dfrac{1}{s_2}\log\|x-y\|, & x \neq y, \\ +\infty, & x = y, \end{cases} \tag{2.5}$$

and by the Newton potential kernel in case $n \geq 3$:

$$h_y(x) = \begin{cases} \dfrac{1}{(n-2)s_n}\dfrac{1}{\|x-y\|^{n-2}}, & x \neq y, \\ +\infty, & x = y. \end{cases} \tag{2.6}$$

More generally, for $a \in \mathbb{R}$ and $y \in \mathbb{R}^n$, the function

$$x \mapsto \|x-y\|^a,$$

is superharmonic on $\mathbb{R}^n$, $n \geq 3$, if and only if $a \in [2-n, 0]$, and harmonic when $a = 2-n$.

We now focus on the Dirichlet problem on the ball $E = B(y, r)$. We consider

$$h_0(r) = -\frac{1}{s_2}\log(r), \qquad r > 0,$$

in case $n = 2$, and

$$h_0(r) = \frac{1}{(n-2)s_n r^{n-2}}, \qquad r > 0,$$

if $n \geq 3$, and let

$$x^* := y + \frac{r^2}{\|y-x\|^2}(x-y)$$

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