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Global Bifurcation of Periodic Solutions with Symmetry

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Preface

The inherent harmony of periodic motions as well as of symmetry has exerted its own fascination, as it seems, ever since the dawn of thought. Today, such a "harmonia mundi" is at least hoped for on just about any possible scale: from elementary particle physics to astronomy.

In search of some harmony let us ask naive questions. Suppose we are given a dynamical system with some built-in symmetry. Should we expect periodic motions which somehow reflect this symmetry? And how would periodicity harmonize with symmetry?

These almost innocent questions are the entrance to a labyrinth of intricacies. Probing only along some fairly safe threads we are lead from dynamics to topology, algebra, singularity theory, numerical analysis, and to some applications. A global point of view will be one guiding theme along our way: we are mainly interested in periodic motions far from equilibrium.

For a method we rely on bifurcation theory, on transversality theory, and on generic approximations. As a reward we encounter known local singularities. As a central new aspect we study the global interaction and interdependence of these local singularities, designing a homotopy invariant. As a result, we obtain an index $\mathcal{X}$ which evaluates only information at stationary solutions. Nonzero $\mathcal{X}$ implies global Hopf bifurcation of periodic solutions with certain symmetries. Putting it emphatically, $\mathcal{X}$ harmonizes symmetry and periodicity . Curiously, $\mathcal{X}$ need not be homotopy invariant. It is one of my favorite speculations that this obstruction may hint at chaotic motions.

Cyclic motions relate to cyclic groups. Phrasing this relation between dynamics and algebra less sloppily: the symmetry of a periodic solution of a dynamical system is related to a cyclic factor within the group of symmetries of that system. Curiously, some period doubling bifurcations relate to the number 2, acting by multiplication on such a cyclic group. The multiplicative order of 2 relates to the number of possibly different indices $\mathcal{X}$ for a given system.

Symmetry, although beautiful, causes numerical difficulties. Basically, groups with irreducible representations of higher dimensions entail higher local singularities which are not very well understood. This is an obstacle to numerical pathfollowing algorithms. We will give a complete list of the easier, lower-dimensional generic bifurcations. Avoiding cyclic loops in the associated global bifurcation diagrams by a suitable homotopy invariant will be a central issue in our theoretical analysis. Both aspects are essential prerequisites for an efficient numerical pathfollowing method in dynamical systems with symmetries.

In real applications, as in real life, the lofty regions of harmony, periodicity, and symmetry are always confronted with the abysmal danger of destabilization. Surprisingly, there are still some applications where periodicity and symmetry is observed. We will concentrate on chemical waves as a model example below, though the theory is general. We obtain rotating waves (spirals) in continuous geometries, and phase-locked oscillations in discrete geometries.

Because it may not at all be easily detected by the reader, let me confess here a guiding principle for this book. Like so many others, I have tried to dismiss difficulty for beauty.

I happily say my thanks to everyone who has helped me. In particular, I would like to mention J. Alexander, G. Auchmuty, T. Bartsch, A. Brandis, S.-N. Chow, R. Cushman, R. Field, S. v. Gils, M. Golubitsky, W. Jäger, P. Kunkel, R. Lauterbach, J. Mallet-Paret, M. Marek, M. Medveď, J.-C. van der Meer, C. Pospiech, J. Sanders, D. Sattinger, R. Schaaf, A. Vanderbauwhede, A. Wagner, J. Yorke, and all those friends who have helped with proofreading. Typesetting the whole manuscript in TEX was a laborious task. It was performed by M. Torterolo with great patience. Finally, I am indebted to Springer-Verlag for an efficient and pleasant cooperation.

Contents

1 Introduction

1.1 The question

We would like to find time-periodic solutions $x(t)$ of a nonlinear autonomous dynamical system

$$\dot{x}(t) = f(\lambda, x(t)), \qquad x \in X := \boldsymbol{R}^N, \quad f \in C^1 \quad . \tag{1.1}$$

In applications, such systems always contain parameters (coefficients). Let $\lambda \in \Lambda := \boldsymbol{R}$ denote one of them. Finding periodic solutions is usually more difficult than finding stationary, i.e. time-independent solutions $x(t) \equiv x_0$. Stationary solutions (λ_0, x_0) satisfy

$$0 = f(\lambda_0, x_0). \tag{1.2}$$

Hopf bifurcation draws conclusions on periodic solutions of (1.1) from information on stationary solutions (1.2); and here and below we mean "nonstationary periodic" when we say periodic.

To describe local Hopf bifurcation suppose for a moment that $f(\lambda, 0) = 0$, for all real λ. Assume that the linearization $D_x f(\lambda, 0)$ at the stationary solution $(\lambda, 0)$ has a pair of simple eigenvalues

$$\lambda \pm i\beta(\lambda), \qquad \beta(\lambda) > 0 \tag{1.3}$$

for small $|\lambda|$. Then at least the linearized equation, at $\lambda = 0$,

$$\dot{y} = D_x f(0, 0) y \tag{1.4}$$

has periodic solutions $y(t)$ of minimal period $2\pi/\beta(0)$. If $\pm i\beta(0)$ are the only purely imaginary eigenvalues of $D_x f(0,0)$, then the local Hopf bifurcation theorem, e.g. [Cra&Rab2], states that (1.1) with $f \in C^2$ has periodic solutions near $\lambda = 0$, $x = 0$. In fact, these periodic solutions form a continous branch and their minimal periods are close to $2\pi/\beta(0)$. Without a parameter λ, i.e. for fixed $\lambda = 0$, such a result could not hold in general.

The result above is called "local", because it only finds periodic solutions in some possibly very small neighborhood of $\lambda = 0$, $x = 0$. Global Hopf bifurcation finds periodic solutions which may be far away from the neighborhood where they originated. The first result in this direction is due to Alexander & Yorke [Ale&Y1], see §1.3 and in particular (1.29) for more details. Global Hopf bifurcation is our main concern here. Of course, global bifurcation implies local bifurcation.

Global as well as local bifurcation results require essentially some change of stability. Let us explain this with our previous example, $f(\lambda, 0) = 0$. Denote

$E(\lambda)$: the number of eigenvalues of $D_x f(\lambda, 0)$ with strictly positive real part, counting algebraic multiplicity. (1.5)

In other words, $E(\lambda)$ is the unstable ("expanding") dimension of the stationary solution $(\lambda, 0)$. Then assumption (1.3) on the crossing of the pair of eigenvalues $\lambda \pm i\beta(\lambda)$ through the imaginary axis implies that $E(\lambda)$ changes by 2 as λ increases through zero. We call this a "change of stability". Our principal goal will be an index $\mathcal{H}$ which evaluates changes of stability in such a way that $\mathcal{H} \neq 0$ implies global Hopf bifurcation.

We are interested in dynamical systems (1.1) with symmetry. Throughout we assume

Γ is a compact Lie group, acting orthogonally on $X := \boldsymbol{R}^N$ by a linear representation ρ. (1.6.a)

In other words:

$$\begin{aligned} \rho : \Gamma &\to O(n) \\ \gamma &\mapsto \rho(\gamma) \end{aligned}$$

is a homorphism from the compact Lie group Γ into the group $O(N)$ of orthogonal $N \times N$-matrices. See e.g. [Bre, Brö&tD, Sat&Wea] for generalities on Lie groups and representations. For practical purposes, we may assume that $\rho(\gamma) = id$ only for $\gamma = id$. This allows us to view Γ as a closed subgroup of $O(N)$. A short-hand notation for the action of Γ is $\gamma x := \rho(\gamma)x$, for $\gamma \in \Gamma$, $x \in X$. To tie up the group Γ with our system (1.1), we require f to be **equivariant** with respect to the action ρ of Γ, i.e.

$$f(\lambda, \gamma x) = \gamma f(\lambda, x), \qquad \text{for all } \gamma \in \Gamma, \quad \lambda \in \boldsymbol{R}, \quad x \in \boldsymbol{R}^N. \tag{1.6.b}$$

Then (1.1) remains unchanged, if we replace x by γx. Thus, if $x(t)$ is a solution of (1.1), then $\gamma x(t)$ is also a solution, regardless which $\gamma \in \Gamma$ we choose. See e.g. [Sat1, Van1] for a reference on bifurcation theory for equivariant f.

If $x(t)$ is a periodic solution of system (1.1), then $\gamma x(t)$ may describe the same trajectory as $x(t)$ for suitably chosen $\gamma \in \Gamma$. In fact γ could leave each point of $x(t)$ fixed, individually. Or γ could leave the periodic orbit $\{x(t) \mid t \in \boldsymbol{R}\}$ fixed, as a set, possibly phase-shifting the individual points on it. In both cases we say that γ belongs to the symmetry of the periodic solution $x(t)$. For more precision see §1.2, definition 1.1. This notion of symmetry leads us to our **principal question**:

How can we find periodic solutions with prescribed symmetry? (1.7)

For linear equivariant equations like (1.4), where $D_x f(0,0)$ has purely imaginary eigenvalues, we might find periodic solution and their symmetry explicitly, knowing the representation of Γ on the eigenspace. For results on local Hopf bifurcation for nonlinear systems with symmetry see e.g. [Go&St1].

We approach question (1.7) from a global point of view. We design an index

$$\mathcal{H}^{\pm d}_{H_0, K_0} \tag{1.8}$$

such that nonzero $\mathcal{H}$ implies global Hopf bifurcation with certain possible symmetries. Again, $\mathcal{H}$ evaluates changes of stability of stationary solutions via purely imaginary eigenvalues in certain representation subspaces of X. For some more details see §1.4. A complete recipe is given in our main results: theorems 2.9 and 2.10 below.

Let us consider a first typical, but simple example: three identical, mutually coupled oscillators. Such examples go back to Turing [Tu]. With $x = (x_0, x_1, x_2)$, $x_j \in \boldsymbol{R}^{\tilde{n}}$, $x \in \boldsymbol{R}^{3\tilde{n}}$ our example may be written as

$$\begin{aligned} \dot{x}_0 &= \tilde{f}(x_0) + (x_2 - 2x_0 + x_1) \\ \dot{x}_1 &= \tilde{f}(x_1) + (x_0 - 2x_1 + x_2) \\ \dot{x}_2 &= \tilde{f}(x_2) + (x_1 - 2x_2 + x_0). \end{aligned} \tag{1.9}$$

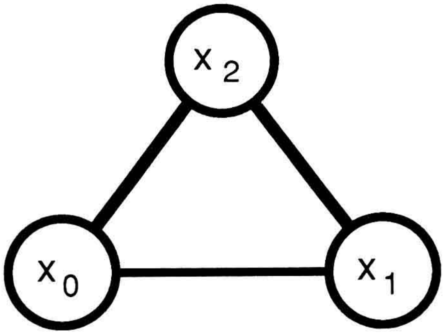

Fig. 1.1 Three coupled oscillators

We suppress the parameter λ, here. In fig. 1.1 we depict system (1.9) as an equilateral triangle. The vertices stand for the oscillators $\dot{x}_0 =$, $\dot{x}_1 =$, $\dot{x}_2 =$, and the sides represent "diffusive " coupling. System (1.9) remains invariant under any permutation of the indices $\{0,1,2\}$; the right hand side is equivariant under $\Gamma := S_3$, the symmetric group (permutations of three elements). From fig. 1.1 we see that S_3 is isomorphic to the dihedral group D_3, the group of orthogonal maps in the plane which leave an equilateral triangle invariant. System (1.9) could oscillate periodically in various ways: homogeneously ($x_0(t) \equiv x_1(t) \equiv x_2(t)$), with reflection symmetry ($x_0(t) \not\equiv x_1(t) \equiv x_2(t)$), with fixed phase-shifts over one third period between adjacent $x_j(t)$, with some other symmetry, or without any noticeable relation between the $x_j(t)$. Answering question (1.7), our index $\mathcal{H}$ will allow us a detailed global analysis of these phenomena, cf. §8.1. The first global results on such rings of coupled oscillators are due to Alexander & Auchmuty [Ale&Au2]. They rely on a topological result on global bifurcation of zeros of mappings with several (two) parameters [Ale1, Ale&Fitz].

Our approach to question (1.7) is more geometrically inclined. Motivated by the "snakes"-paper of Mallet-Paret & Yorke [M-P&Y1,2] we use generic, but equivariant approximations to the original problem (1.1). This will have the advantage that only a few types of bifurcations occur, and that global bifurcation diagrams can be understood systematically. We discuss this in §1.5 and, in excessive detail, in §§3,5-7,10. In [M-P&Y1,2], only the case of no symmetry, $\Gamma = \{id\}$, was considered. Another root of our approach is the elegant geometric treatment of local equivariant Hopf bifurcation by Golubitsky & Stewart [Go&Sch&St, Go&St1]. It inspired the very question (1.7), as well as our definition of symmetry of a periodic solution, and is behind the scene of most of our technical set-up.

Why should anyone be interested in a question like (1.7)? Our motivation is both "pure" and "applied". Symmetry prevails in many applied problems, e.g. oscillations in networks, in fluid dynamics, and in chemical reaction diffusion systems. A spectacular example are the rotating spirals in the Belousov-Zhabotinskii reaction, see fig. 1.2.

We devote §8 to such applications. Another "applied" goal is the development of quick, flexible tests which detect oscillations and give some indication of their spatio-temporal form in large distributed systems. Paradoxically, global results apply more easily than local results (but do not allow conclusions on stability, direction of bifurcation, etc.). As a "pure"

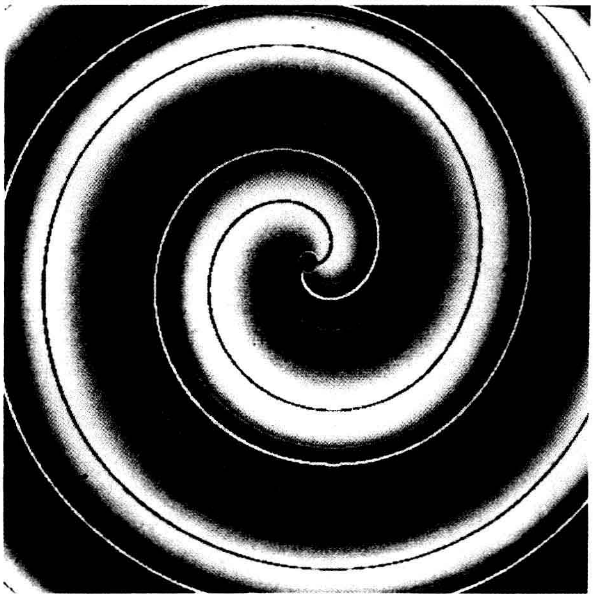

Fig. 1.2 A (clockwise) rotating spiral wave, courtesy of [Mü&Ple&Hess].

consequence we obtain local bifurcating branches for situations which could not be treated in [Go&St1], see theorem 9.1.

But local bifurcations, local singularities have been studied for quite a while now, even in equivariant settings. Our analysis adds a significant global feature: we investigate the interplay of these local singularities in global bifurcation diagrams. We believe that this global feature can and should be incorporated into other contexts as well. The problem of global Hopf bifurcation with symmetry just serves us as a model case.

Understanding the interplay of local singularities in global bifurcation diagrams usually uncovers some topological relations and restrictions, expressed by homotopy invariant indices. Knowing these global restrictions, as well as the basic local singularities, is in turn a prerequisite to the design of a successful numerical homotopy method for concrete applications. The simplest example is the monitoring of signs of determinants of the linearization, i.e. of Brouwer degree, to detect stationary bifurcation points; see e.g. [Deu&Fie&Kun]. This closes the circle of "pure" and "applied" motivations.

1.2 Symmetry of periodic solutions

Let us pin down what we mean by the symmetry of a periodic solution $x(t)$ of the Γ-equivariant differential equation (1.1). First we have to discuss "symmetries" of points $x \in X$. Given $x \in X$ the **isotropy** group Γ_x of x is defined as

$$\Gamma_x := \{\gamma \in \Gamma \mid \gamma x = x\}. \tag{1.10}$$

For example, consider the coupled oscillator system (1.9). If $x = (x_0, x_1, x_2)$ with $x_0 = x_1 = x_2$ then $\Gamma_x = \Gamma = S_3$. If $x_0 \neq x_1 = x_2$, then $\Gamma_x = \{id, (1\ 2)\} =: \langle(1\ 2)\rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Conversely, given a subgroup K of Γ we may define the **fixed point subspace** X^K of X by

$$X^K := \{x \in X \mid Kx = x\}. \tag{1.11}$$

So X^K consists of all elements x of X with isotropy Γ_x at least K. In the example (1.9) we have $x \in X^\Gamma$ iff $x_0 = x_1 = x_2$, and $x \in X^{\langle(1\ 2)\rangle}$ iff $x_1 = x_2$.

This last example shows that X^K may actually contain points x with $\Gamma_x > K$. Throughout, we are interested in this typical case of a non-free group action, i.e. the conjugacy class of Γ_x may depend on the choice of $x \neq 0$.

For solutions of (1.1) one would like to know Γ_x. The significant property of the linear subspaces X^K, on the other hand, is their flow invariance:

$$x(0) \in X^K \quad \text{implies} \quad x(t) \in X^K, \text{ for all } t. \tag{1.12}$$

Indeed, $x \in X^K$ implies $\dot{x} \in X^K$, because

$$K\dot{x} = Kf(\lambda, x) = f(\lambda, Kx) = f(\lambda, x) = \dot{x}.$$

Now consider a periodic solution $x(t)$ of (1.1) with minimal period $p > 0$. Let $C := \{x(t) \mid t \in \boldsymbol{R}\} \subset X$ denote the trajectory of $x(t)$. Then two relevant groups come to mind:

$$H := \{\gamma \in \Gamma \mid \gamma C = C\} \tag{1.13.a}$$

$$K := \Gamma_{x(t)} = \{\gamma \in \Gamma \mid \gamma x(t) = x(t)\}. \tag{1.13.b}$$

Note that $\Gamma_{x(t)}$ is in fact independent of t because, by flow invariance of the spaces X^K, $x(0) \in X^{\Gamma_{x(t)}}$ and $x(t) \in X^{\Gamma_{x(0)}}$, i. e. $\Gamma_{x(0)} \geq \Gamma_{x(t)}$ and $\Gamma_{x(t)} \geq \Gamma_{x(0)}$. Thus K is well-defined. Obviously, K is a subgroup of the closed group H. For any $h \in H$, $x(t) \in C$, we have

$$h\ x(t) \quad = \quad x(t + \Theta(h)\ p). \tag{1.13.c}$$

Note that $\Theta(h) \in \boldsymbol{R}/\boldsymbol{Z}$ is defined independently of t. In fact $hx(t)$ solves the same differential equation (1.1) as $x(t)$ and the trajectories coincide as sets, by (1.13.a). Thus $hx(t)$ coincides with $x(t)$, up to a phase shift.

The obviously continuous map

$$\begin{aligned} \Theta: \quad & H \to \boldsymbol{R}/\boldsymbol{Z} \\ & h \mapsto \Theta(h) \end{aligned} \tag{1.14}$$

from H to the (additive) group $\boldsymbol{R}/\boldsymbol{Z}$ is a homomorphism. Indeed

$$\begin{aligned} x(t + \Theta(h_1 h_2) \cdot p) &= h_1 h_2\ x(t) = h_1\ x(t + \Theta(h_2) \cdot p) = \\ &= x(t + (\Theta(h_1) + \Theta(h_2)) \cdot p), \qquad i.e. \\ \Theta(h_1\ h_2) &\equiv \Theta(h_1) + \Theta(h_2) \quad (mod\ \boldsymbol{Z}), \end{aligned}$$

because p is the minimal period of $x(t)$. By definition, $ker\Theta = K$. By the homomorphism theorem [vdW, Lang], K is a closed normal subgroup of the Lie group H and

$$H/K \quad \cong \quad im\Theta \tag{1.15}$$

may be viewed as a closed subgroup of $\mathbb{R}/\mathbb{Z}$, cf. [Brö&tD, §1.4]. Let

$$\begin{aligned} \mathbb{Z}_n &:= \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \leq \mathbb{R}/\mathbb{Z}, \text{ for } n < \infty \\ \mathbb{Z}_\infty &:= \mathbb{R}/\mathbb{Z} \end{aligned} \tag{1.16.a}$$

denote the closed additive subgroups of $\mathbb{R}/\mathbb{Z}$. The cyclic groups $\mathbb{Z}_n$ should not be mixed up with the isomorphic groups

$$\begin{aligned} \mathbb{Z}(n) &:= \{0, 1, \ldots, n-1\} = \mathbb{Z}/n\mathbb{Z}, \\ \mathbb{Z}(\infty) &:= \mathbb{Z}. \end{aligned} \tag{1.16.b}$$

With this notation, (1.15) implies that

$$H/K \cong \begin{cases} \mathbb{Z}_n & \text{for some } n < \infty, \quad \text{or} \\ \mathbb{Z}_\infty \end{cases} \tag{1.17}$$

1.1 Definition :

Let $x(t)$ be a periodic solution of system (1.1) with minimal period $p > 0$. We call the triple (H, K, Θ), defined by (1.13.a-c) above, the **symmetry** *of $x(t)$.*

Referring to (1.17) above, we call $x(t)$ a

concentric wave	*if*	$H = K$
discrete wave	*if*	$H/K \cong \mathbb{Z}_n, \quad 1 \leq n < \infty$
rotating wave	*if*	$H/K \cong \mathbb{Z}_\infty$.

Let $C \subseteq X$ be a set of stationary solutions of $f(\lambda, \cdot)$, i.e.

$$f(\lambda, x) = 0 \qquad \text{for all } x \in C. \tag{1.18.a}$$

We call C a **frozen wave**, *if there exists $x_0 \in C$ and subgroups $K := \Gamma_{x_0} \leq H \leq \Gamma$ such that the following two conditions hold:*

$$C = H \cdot x_0 \tag{1.18.b}$$

$$K = \Gamma_{x_0} \text{ is normal in } H \text{ and } H/K \cong \mathbb{Z}_\infty. \tag{1.18.c}$$

We call the triple $(H, K, \pm\Theta)$ the **symmetry** *of the frozen wave C, if $\Theta : H \to \mathbb{R}/\mathbb{Z}$ is any surjective homomorphism with kernel K.*

We comment on definition 1.1. First of all, it seems redundant to include $K = ker\Theta$ explicitly in the triple (H, K, Θ) which defines symmetry. Indeed, H and Θ alone would suffice. Discussing secondary bifurcations it will be convenient though to nevertheless keep track of K explicitly.

The terms concentric wave and rotating wave refer to the phenomenology of the Belousov-Zhabotinskii reaction in a petri dish, $\Gamma \geq SO(2)$. Concentric waves (alias target patterns), corresponding to $H = K = SO(2)$, are observed in form of circular rings propagating radially outwards in a time periodic fashion. For a snapshot of a rotating wave see fig. 1.2. Examples of discrete waves for the discrete symmetry of a triangular Turing ring, fig. 1.1, are discussed below. Note that concentric waves are somewhat degenerate examples of discrete waves.

In the symmetry $(H, K, \pm\Theta)$ of a frozen wave, the homomorphism Θ is determined only up to a sign. Indeed, Θ induces an isomorphism $H/K \to \boldsymbol{R}/\boldsymbol{Z}$, and the only continuous automorphisms of $\boldsymbol{R}/\boldsymbol{Z}$ are given by multiplication with ± 1.

Condition (1.18.c) suggests that frozen waves are a pendant to rotating waves. Indeed, let $\mathcal{R}$ be the infinitesimal generator of the action of H/K on X^K. In detail: we represent this action by orthogonal matrices, and obtain an isomorphism

$$\begin{aligned} \iota: \quad \mathbb{Z}_\infty &\to H/K \\ t &\mapsto exp(\mathcal{R}t). \end{aligned} \tag{1.19}$$

For some real α, consider the transformation

$$y(t) \quad = \quad exp(-\alpha\mathcal{R}t)\ x(t) \tag{1.20}$$

on X^K. Then y solves the equation

$$\dot{y}(t) = -\alpha\mathcal{R}\ y(t) + f(\lambda, y(t)) =: \hat{f}(\lambda, y(t)). \tag{1.21}$$

Choosing $\alpha = 1/p$, it turns out that $x(t)$ is a rotating wave for f iff $H \cdot x(0)$ is a frozen wave for $\hat{f}$. The transformation (1.20) tells us that a rotating wave $x(t)$ "freezes", if viewed in a suitable rotating coordinate frame.

Conversely, let us start from a frozen wave $x \in C$ with symmetry $(H, K, \pm\Theta)$. Then the transformation (1.20) yields a rotating wave $y(t)$ with symmetry (H, K, Θ) or $(H, K, -\Theta)$, depending on the sign of α. Viewing this as a perturbation result we may say that a rotating wave freezes and then starts rotating in the opposite direction, cf. definition 5.3 of a freezing, and theorem 5.11.

Viewed still differently, (1.20) and (1.21) tell us that $(\lambda, x(t))$ is a rotating wave if and only if $x_0 = x(0)$ with $\mathcal{R}x_0 \neq 0$ solves

$$0 = -\alpha\mathcal{R}x_0 + f(\lambda, x_0) \tag{1.21$'$}$$

for some $\alpha \neq 0$. On the other hand, Hx_0 is a frozen wave if and only if x_0 with $\mathcal{R}x_0 \neq 0$ solves $(1.21)'$ for $\alpha = 0$.

Let us reinterpret symmetry of periodic solutions in an operator setting which is commonly used in global Hopf bifurcation. We rescale the minimal period p of $x(t)$ to 1, defining

$$\xi(\tau) := x(p\tau) \quad . \tag{1.22}$$

Then $x(t)$ solves (1.1) iff ξ solves

$$F(f, p, \lambda, \xi) := -\frac{1}{p}\dot{\xi} + f(\lambda, \xi) = 0 \quad . \tag{1.23}$$

Denoting the Banach spaces of continuous resp. once continuously differentiable functions with (not necessarily minimal) period 1 by $\tilde{C}^0$ resp. $\tilde{C}^1$, we may view $F(f, \cdot, \cdot, \cdot)$ for fixed f as a map

$$F(f, \cdot, \cdot, \cdot): \quad \boldsymbol{R}^+ \times \boldsymbol{R} \times \tilde{C}^1 \ \to\ \tilde{C}^0 \ . \tag{1.24}$$

Fixing also p, λ, the map $F(f, p, \lambda, \cdot)$ is equivariant with respect to the action $\tilde{\rho}$ of $\tilde{\Gamma} := \Gamma \times S^1$ on $\tilde{\xi} \in \tilde{C}^0$ or $\tilde{C}^1$ defined by

$$(\tilde{\rho}(\gamma, \vartheta)\tilde{\xi})(\tau) \quad := \quad \rho(\gamma)\tilde{\xi}(\tau - \vartheta) \; , \tag{1.25}$$

where we write S^1 for the additive group $\boldsymbol{R}/\mathbb{Z}$.

We claim that x has symmetry (H, K, Θ) iff ξ, defined by (1.22), has isotropy

$$\tilde{\Gamma}_\xi = H^\Theta := \{(h, \Theta(h)) \mid h \in H\}. \tag{1.26}$$

As before, it is understood that $K := ker\Theta$. To prove our claim, we follow the reasoning in [Go&St1, §6]. Applying the definition of symmetry of x, it is sufficient to show that $\tilde{\Gamma}_\xi = \tilde{H}^{\tilde{\Theta}}$ for some subgroup $\tilde{H}$ of Γ and some homomorphism $\tilde{\Theta} : \tilde{H} \to S^1$. Let $\pi : \Gamma \times S^1 \to \Gamma$ denote projection onto the first factor and define $\tilde{H} := \pi(\tilde{\Gamma}_\xi)$. Then $\tilde{\Gamma}_\xi \cap ker\pi = \{id\}$, because ξ has minimal period 1. Thus $\tilde{\Gamma}_\xi \cong \tilde{H}$, and we may hence write $\tilde{\Gamma}_\xi$ as $\tilde{H}^{\tilde{\Theta}}$ as was claimed above. Following [Go&St1, §6] we call H^Θ a twisted subgroup of $\Gamma \times S^1$ with twist Θ.

Fixing an isomorphism from $\mathbb{Z}_n$ to H/K, we may represent the twist Θ by an integer $(mod\ n)$. Indeed, let hK generate $H/K \cong \mathbb{Z}_n$ (assuming $n < \infty$) and fix ι to be given by

$$\begin{aligned} \iota : \quad & \mathbb{Z}_n \to H/K \\ & \tfrac{1}{n} \mapsto hK. \end{aligned} \tag{1.27}$$

Then

$$\Theta(h) \quad = \quad (\Theta \circ \iota)(\tfrac{1}{n}) \quad = \quad \Theta^* \,/\, n$$

for some $\Theta^* \in \mathbb{Z}(n)$. We will frequently identify h with $1/n$ and Θ with $\Theta^* \in \mathbb{Z}(n)$, writing

$$\Theta(h) = \Theta \cdot h. \tag{1.28}$$

Using the isomorphism (1.19) instead of (1.27), the case $n = \infty$ is treated similarly. Representing Θ by integers is particularly convenient at secondary bifurcations of periodic solutions, where Θ may change, cf. §5.

We illustrate our symmetry terminology with the triangle of coupled oscillators (1.9), see fig. 1.1. Concentric waves, e.g., are periodic solutions $x(t)$ with $x_0(t) \equiv x_1(t) \equiv x_2(t)$. Their symmetry is $(H, K, \Theta) = (\Gamma, \Gamma, \Theta)$ where $\Gamma = S_3$. They satisfy $\dot{x}_0 = \tilde{f}(x_0)$, and diffusive coupling can be ignored altogether. Another example is given by $x_0(t) \not\equiv x_1(t) \equiv x_2(t)$ with $H = K = \langle(1\ 2)\rangle$, $\Theta = 0$. Such solutions satisfy

$$\begin{aligned} \dot{x}_0 &= \tilde{f}(x_0) + 2(x_1 - x_0) \\ \dot{x}_1 &= \tilde{f}(x_1) + (x_0 - x_1) \end{aligned}$$

and represent two asymmetrically coupled oscillators. A discrete wave could have symmetry $H = \langle(1\ 2)\rangle, K = \{id\}$, and $\Theta = 1$, which means $x_2(t) = x_1(t - \frac{p}{2})$ and $x_0(t) = x_0(t - \frac{p}{2})$. Such solutions are sometimes called standing waves. Another type of discrete waves satisfies $H = \langle(0\ 1\ 2)\rangle, K = \{id\}$, and $\Theta = 1$, which corresponds to

$$x_2(t) \equiv x_1(t - \frac{p}{3}) \equiv x_0(t - \frac{2p}{3}),$$

i.e. to fixed phase-differences between adjacent cells. Applying $(1\ 2) \in S_3$ to this solution we obtain a discrete wave with $\Theta = 2 \equiv -1\ (mod\ 3)$, i.e. rotation in the opposite direction. For specific examples of rotating and frozen waves see §8.2.

1.3 Some references

The literature on bifurcation problems is vast. We give some standard references to the field. Then we follow some of the threads to global bifurcation, concentrating on Hopf bifurcation. A more detailed attempt to put our results in perspective has to be postponed to §9. As a general reference to local bifurcation theory we mention the books by Chow & Hale [Chow&Ha], Golubitsky & Schaeffer [Go&Sch], Guckenheimer & Holmes [Gu&Ho], Iooss & Joseph [Io&Jo], as well as parts of Arnold [Arn3, ch.6], and Smoller [Smo, ch.13]. Bifurcations for iterates of maps are discussed e.g. in [Io]. Bifurcation theory for zeros of maps, viz. stationary solutions with several parameters, is known as singularity theory or catastrophe theory, see e.g. [Arn4, Arn&G-Z&Var, Go&Gui, Thom].

More specifically, local Hopf bifurcation is named after E. Hopf. In [Hopf], 1942, he proves the result which we have discussed in §1.1, assuming $x \in \mathbb{R}^N$ and f analytic. His main motivation, though, was hydrodynamics. Hopf himself mentions Poincaré, who has considered the planar analytic Hamiltonian case being mainly motivated by periodically forced systems in celestial mechanics, cf. [Poi, ch.XXX], 1899. The general planar case was discussed extensively by Andronov and coworkers since 1929, see e.g. [And&Chai, And&Leo&Gor&Mai] and the note in [Arn3, p.271]. In 1977 a proof covering the infinite-dimensional case was given by Crandall & Rabinowitz [Cra&Rab2] in an analytic semigroup C^2-setting. They just relied on the implicit function theorem. Other modern accounts of local Hopf bifurcation, three of them based on center manifolds, are given e.g. in the books of Chow & Hale [Chow&Ha], Hassard & Kazarinoff & Wan [Has&Kaz&Wan], Iooss & Joseph [Io&Jo], and Marsden & McCracken [Mars&McCr].

The first global bifurcation result, concerning stationary solutions, is due to Rabinowitz [Rab]. Returning to the setting $f(\lambda,0) = 0$ with unstable dimension $E(\lambda)$, as in (1.5), a version may be phrased as follows. If $E(\lambda)$ changes by an odd number, as λ increases from $-\infty$ to $+\infty$, then an unbounded continuum of stationary solutions bifurcates from the trivial solution. The proof relies on degree theory, and we give a subjective version of it in §3; see also [Chow&Ha, §5.8] and [Smo, ch.13].

As we have mentioned above, the first result on global Hopf bifurcation without symmetry is due to Alexander & Yorke [Ale&Y1]; see also Ize [Ize1]. They both introduce period p explicitly as a parameter. In the above setting, suppose $D_x f(0,0)$ is nondegenerate, and $D_x f(\lambda, 0)$ has some purely imaginary eigenvalues for $\lambda = 0$ but not for small $0 < |\lambda| \leq \varepsilon$. Assuming that

$$\frac{1}{2}(E(\varepsilon) - E(-\varepsilon)) \quad \text{is} \quad \text{odd}, \tag{1.29}$$

they obtain a global bifurcating continuum $\hat{C}$ of periodic solutions, by topological arguments involving stable homotopy theory. "Continuum" refers to the triple (p, λ, ξ), and "global" means that $\hat{C}$ is unbounded or returns to some other bifurcation point on the trivial branch. Using Fuller index [Ful], Chow & Mallet-Paret & Yorke [Chow&M-P&Y1] later relaxed condition (1.29) to

$$\frac{1}{2}(E(\varepsilon) - E(-\varepsilon)) \quad \neq \quad 0. \tag{1.29$'$}$$

These results have one obvious and one subtle drawback. Obviously, we might not want to call $\hat{C}$ "global", if it remains bounded and just terminates at some other Hopf bifurcation point. It is a more subtle aspect to construct examples of continua in (p, λ, ξ) which are unbounded, though λ, ξ and minimal periods remain bounded. A concrete example for this important subtlety was constructed by Alligood & Mallet-Paret & Yorke [All&M-P&Y1],

cf. §3.4 and fig. 3.3 below. This is possible because p in the operator setting (1.23) does not necessarily stand for minimal period. In fact, if (p, λ, ξ) is a solution then (kp, λ, ξ^k) is likewise a solution, if we define

$$\xi^k(\tau) := \xi(k\tau).$$

For a detailed discussion see §§3 and 9.3.

Both drawbacks have been circumvented at the expense of introducing the notion of "virtual periods", cf. definition 1.2 below and §4. For generic nonlinearities $f(\lambda, x)$ the drawbacks were fully remedied by Mallet-Paret & Yorke [M-P&Y1,2], who follow continua ("snakes") in (λ, x) and simultaneously keep track of minimal period. Virtual periods, as introduced by Chow, Mallet-Paret, Yorke [M-P&Y2, Chow&M-P&Y2], arise if one approximates f in (1.1) by generic nonlinearities. Following [Fie2], we give a detailed outline of this no-symmetry theory in §3 because it will be basic to our symmetry results.

Including symmetry, the books of Golubitsky, Schaeffer, Stewart [Go&Sch, Go&Sch&St], Sattinger [Sat1,2], and Vanderbauwhede [Van1,5] treat local bifurcations extensively. For a detailed study of local symmetry-breaking in elliptic equations see [Smo&Wal-3, Van3, Van5]. Concerning local Hopf bifurcation with symmetry we have mentioned [Go&St1]. Rotating waves were also discussed, e.g., in [Au, Sche, Van2].

Global results are few in number. Globally-minded bifurcation of stationary solutions with symmetry was achieved by Cerami [Cer], Cicogna [Cic], and Pospiech [Pos1-3]. They all essentially pick a subgroup K of Γ and proceed along the global result of Rabinowitz [Rab] within the f-invariant subspace X^K. We could imitate this for periodic solutions, because X^K is invariant under the flow (1.1). In X^K the no-symmetry theorems from [Ale&Y1, Chow&M-P&Y1, Ize1, Ize2, Fie2] readily apply. For concentric waves ($H = K$, cf. §1.2) this approach is certainly appropriate. But it is not for $H > K$: all information on H and the action of Θ along the periodic solution will be lost completely. We are aware of only two previous results on global Hopf bifurcation with symmetry, which address this problem. Both are due to Alexander & Auchmuty: see [Ale&Au1] for rotating waves in a reaction diffusion system, and [Ale&Au2] for discrete waves in coupled oscillators.

However, these results are obtained via an operator setting similar to (1.23). They prescribe some symmetry $(\tilde{H}, \tilde{K}, \tilde{\Theta})$ for the periodic solutions $x(t)$, roughly as in definition 1.1, i.e.

$$h\, x(t) \quad = \quad x(t + \tilde{\Theta}(h)\tilde{p}), \quad \text{for all } h \in \tilde{H}, \tag{1.30}$$

but they do not know whether $\tilde{p}$ is the minimal period p of $x(t)$, or just some multiple kp of it. This way they obtain $H \geq \tilde{H}$, but no information on Θ. In fact one can only conclude that

$$\Theta(h) \quad \equiv \quad k \cdot \tilde{\Theta}(h) \qquad (mod\ 1) \tag{1.31}$$

for some unknown k. For example, if $im\tilde{\Theta} \cong \tilde{H}/\tilde{K}$ is finite then Θ may be identically 0, picking $k = |\tilde{H}/\tilde{K}|$.

We are aiming at results which keep control of Θ and, at the same time, remedy the two drawbacks of the topology approach mentioned earlier. We will return to a comparison with the results of Alexander & Auchmuty in §9.4.

1.4 Virtual answers

Our notion of virtual symmetry is the key to our main results, summarized in theorems 2.9 and 2.10. "Mostly" virtual symmetry will coincide with symmetry, cf. definition 1.1. Virtual symmetry is defined as follows.

1.2 Definition :
Let $x = x(t)$ be a stationary or a periodic solution of

$$\dot{x}(t) = f(\lambda, x(t)) \quad . \tag{1.1}$$

We call $q > 0$ a **virtual period** *of x, and $(\hat{H}, \hat{K}, \hat{\Theta})$ a* **virtual symmetry** *of x, if there exists a solution y of the linearized equation*

$$\dot{y}(t) = D_x f(\lambda, x(t))\ y(t) \tag{1.32}$$

such that the pair $(x(t), y(t))$ has minimal period q and symmetry $(\hat{H}, \hat{K}, \hat{\Theta})$ in the sense of definition 1.1; in particular

$$\begin{aligned} \rho(h)\ x(t) &= x(t + \hat{\Theta}(h)\ q) \\ \rho(h)\ y(t) &= y(t + \hat{\Theta}(h)\ q) \end{aligned} \tag{1.33}$$

for all $h \in \hat{H}$.
Similarly, suppose $f(\lambda, x) = 0$ and $y \in ker D_x f(\lambda, x)$ is such that the pair (x, y) lies on a frozen wave $\hat{H} \cdot (x, y)$ with symmetry $(\hat{H}, \hat{K}, \pm\hat{\Theta})$ in the sense of definition 1.1, i.e. $\hat{K} = \Gamma_{(x,y)} := \Gamma_x \cap \Gamma_y$ is the isotropy of the pair (x, y). Then we also call $(\hat{H}, \hat{K}, \pm\hat{\Theta})$ a virtual symmetry of x.

The notion of virtual period is due to Chow, Mallet-Paret, and Yorke, see [M-P&Y2, Chow&M-P&Y2]. To be precise we should call q a "virtual period of x with respect to $f(\lambda, \cdot)$" etc., but for brevity we don't. Also, λ is fixed in definition 1.2 and we might as well omit it.

Note that the minimal period $p > 0$ and the symmetry (H, K, Θ) of a periodic solution $x(t)$ are always a virtual period and a virtual symmetry of x, just putting $y \equiv 0$ or also $y \equiv \dot{x}$. Suppose $\dot{x}$ and its scalar multiples are the only periodic solutions of the variational equation (1.32). Then the minimal period is the only virtual period, and the symmetry is the only virtual symmetry of x. In particular this is the case for hyperbolic periodic solutions, i.e. for most "typical" periodic solutions. In general x may have several, but finitely many, virtual periods and virtual symmetries. For stationary solutions x_0 the above remarks apply analogously. Note however, that a stationary solution x_0 has some virtual period and some virtual symmetry iff $D_x f(\lambda, x_0)$ has some purely imaginary nonzero eigenvalues, cf. lemma 4.8. Otherwise (x, y) is necessarily stationary and its "minimal period" q is not positive. For a thorough discussion of virtual symmetry see §4.

Next we describe at least the general flavor of our main results, theorems 2.9 and 2.10. For Γ-equivariant systems (1.1) we first fix any two closed subgroups $K_0 \leq H_0$ of Γ such that K_0 is normal in H_0 and $H_0/K_0 \cong \mathbb{Z}_n$ is cyclic, $n \leq \infty$; the notation follows (1.16.a) above. A priori, these subgroups H_0, K_0 need not correspond to any symmetry (H, K, Θ) of any periodic solution at all. Next we pick a certain subset d of $\mathbb{Z}(n)$, a so-called "binary orbit", cf. definition 2.4 and table 2.2. The set d describes some maximal orbit in $\mathbb{Z}(n)$ under iterated