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# INDUCTION IN GEOMETRY

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Л. И. Головина и И. М. Яглом

# ИНДУКЦИЯ В ГЕОМЕТРИИ

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L. I. Golovina and I. M. Yaglom

# INDUCTION IN GEOMETRY

Translated from the Russian
by
Leonid Levant

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*На английском языке*

# CONTENTS

# PREFACE TO THE ENGLISH EDITION

This little book is intended primarily for high school pupils, teachers of mathematics and students in teachers training colleges majoring in physics or mathematics. It deals with various applications of the method of mathematical induction to solving geometric problems and was intended by the authors as a natural continuation of I. S. Sominsky's booklet "The Method of Mathematical Induction" published (in English) by Mir Publishers in 1975. Our book contains 38 worked examples and 45 problems accompanied by brief hints. Various aspects of the method of mathematical induction are treated in them in a most instructive way. Some of the examples and problems may be of independent interest as well.

The book is based on two lectures delivered by Professor I. M. Yaglom to the School Mathematical Circle at the Moscow State University.

The present English edition of the book differs from the Russian original by the inclusion of further examples and problems, as well as additional relevant information on some of the latest achievements in mathematics. It is supplied with a new bibliography in a form convenient for English readers.

L. I. Golovina
I. M. Yaglom

# INTRODUCTION: THE METHOD OF MATHEMATICAL INDUCTION

Any reasoning involving passage from particular assertions to general assertions, which derive their validity from the validity of the particular assertions, is called *induction*. The *method of mathematical induction* is a special method of mathematical proof which enables us to draw conclusions about a general law on the basis of particular cases. The principle of this method can be best understood from examples. So let us consider the following example.

EXAMPLE 1. Determine the sum of the first $n$ odd numbers

$$1 + 3 + 5 + \ldots + (2n - 1).$$

*Solution.* Denoting this sum by $S(n)$, put $n = 1, 2, 3, 4, 5$. We shall then have

$$S(1) = 1,$$
$$S(2) = 1 + 3 = 4,$$
$$S(3) = 1 + 3 + 5 = 9,$$
$$S(4) = 1 + 3 + 5 + 7 = 16,$$
$$S(5) = 1 + 3 + 5 + 7 + 9 = 25.$$

We note that for $n = 1, 2, 3, 4, 5$ the sum of the $n$ successive odd numbers is equal to $n^2$. We cannot conclude at once from this that it holds for any $n$. Such a conclusion "by analogy" may sometimes turn out to be erroneous. Let us illustrate our assertion by several examples.

Consider numbers of the form $2^{2^n} + 1$. For $n = 0, 1, 2, 3, 4$ the numbers $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, $2^{2^4} + 1 = 65\,537$ are primes. A notable seventeenth century French mathematician, P. Fermat, conjectured that all numbers of this form are primes. However, in the eighteenth century another great scientist L. Euler, an academician of Petersburg, discovered that

$$2^{2^5} + 1 = 4\,294\,967\,297 = 641 \times 6\,700\,417$$

is a composite number.

Here is another example. A famous seventeenth century German mathematician, one of the creators of higher mathematics, G. W. Leibnitz, proved that for every positive integer $n$, the number $n^3 - n$ is divisible by 3, $n^5 - n$ is divisible by 5, $n^7 - n$ is divisible by 7 *. On the basis of this, he was on the verge of conjecturing that for every odd $k$ and any natural number $n$, $n^k - n$ is divisible by $k$ but soon he himself noticed that $2^9 - 2 = 510$ is *not divisible* by 9.

A well-known Soviet mathematician, D. A. Grave, once lapsed into the same kind of error, conjecturing that $2^{p-1} - 1$ is not divisible by $p^2$ for any prime number $p$. A direct check confirmed this conjecture for all primes $p$ less than one thousand. Soon, however, it was established that $2^{1092} - 1$ is divisible by $1093^2$ (1093 is a prime), i. e. Grave's conjecture turned out to be erroneous.

Let us consider one more convincing example. If we evaluate the expression $991n^2 + 1$ for $n = 1, 2, 3, \ldots$, i. e. for a succession of whole numbers, we shall never get a number which is a perfect square, even if we spend many days or even years on the problem. But we would be mistaken concluding from this that *all* numbers of this kind are not squares, since in fact among the numbers of the form $991n^2 + 1$ there are squares; only the least value of $n$ for which the number $991n^2 + 1$ is a perfect square is very large. Here is the number:

$$n = 12\,055\,735\,790\,331\,359\,447\,442\,538\,767.$$

All these examples must warn the reader against groundless conclusions drawn by analogy.

Let us now return to the problem of calculating the sum of the first $n$ odd numbers. As is clear from what was said above, the formula

$$S(n) = n^2 \tag{1}$$

cannot be considered as proven, whatever the number of values of $n$ it is checked for, since there is always the possibility that somewhere beyond the range of the cases considered the formula ceases to be true. To make sure that the formula (1) is valid

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* See, for example [21].

*for all n* we have to prove that however far we move along the natural number sequence, we can never pass from the values of $n$ for which the formula (1) is true to values for which it no longer holds.

Hence, *let us assume that for a certain number n our formula is true, and let us try to prove that it is then also true for the next number*, $n + 1$.

Thus, we assume that

$$S(n) = 1 + 3 + 5 + \ldots + (2n - 1) \doteq n^2.$$

Let us compute

$$S(n + 1) = 1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1).$$

By the hypothesis, the sum of the first $n$ terms in the right-hand member of the last equality is equal to $n^2$, hence

$$S(n + 1) = n^2 + (2n + 1) = (n + 1)^2.$$

Thus, by assuming that the formula $S(n) = n^2$ is true for some natural number $n$, we were able to prove its validity for the next number $n + 1$. But we checked above that the formula is true for $n = 1, 2, 3, 4, 5$. Consequently, it will also be true for the number $n = 6$ which follows 5, and then it holds for $n = 7$, for $n = 8$, and for $n = 9$, and so on. Our formula may now be considered proven for *any* number of terms. This method of proof is called the *method of mathematical induction*.

Thus, *a proof by the method of mathematical induction consists of the following two parts*:

1°. *A check that the assertion is valid for the least value of n for which it remains meaningful**;

2°. *A proof that if the assertion is valid for some arbitrary natural number n, then it is also valid for* $n + 1$.

The examples considered above convince us of the necessity of the second part of the proof. The first part of the reasoning is clearly also necessary. It must be emphasized that a proof by the method of mathematical induction definitely requires proof of both parts, 1° and 2°.

A proof that if some proposition is valid for some number $n$, then it is also valid for the number $n + 1$, on its own, is not enough, since it may turn out that this assertion is not true *for*

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* It goes without saying that this value of $n$ is not necessarily equal to unity. For instance, any assertion concerning general properties of arbitrary $n$-gons only makes sense for $n \geqslant 3$.

*any* integral value of $n$. For example, if we assume that a certain integer $n$ is equal to the next natural number, i.e. that $n = n + 1$, then, by adding 1 to each side of this equality, we obtain $n + 1 = n + 2$, i.e. the number $n + 1$ is also equal to the next number following it. But it does not follow from this fact that the assertion stated is valid for all $n$ – on the contrary: it is not true for any whole number.

The method of mathematical induction is not necessarily applied strictly in accordance with the above scheme. For example, we sometimes assume that the conjecture under consideration is valid, say, for *two* successive numbers $n - 1$ and $n$, and have to prove that in this case it is valid for the number $n + 1$ as well. In this case we must begin our reasoning with a check in order to verify that the conjecture is true for the first two values of $n$, for instance, for $n = 1$ and $n = 2$ (see Examples 17, 18 and 19). As a second step in our reasoning, we sometimes prove the validity of the conjecture for a certain value of $n$, assuming its validity for *all* natural numbers $k$ *less* than $n$ (see Examples 7, 8, 9, 16).

Let us consider some more examples of applying the method of mathematical induction. The formulas obtained in solving them will be used later on.

EXAMPLE 2. Prove that the sum of the first $n$ natural numbers (let us denote it by $S_1(n)$) is equal to $\frac{n(n+1)}{2}$, i.e.

$$S_1(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \qquad (2)$$

*Solution.* 1°. $S_1(1) = 1 = \frac{1(1+1)}{2}$.

2°. Suppose that

$$S_1(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Then

$$S_1(n+1) = 1 + 2 + 3 + \ldots + n + (n+1) =$$
$$= \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)[(n+1)+1]}{2},$$

which completely proves the assertion.

EXAMPLE 3. Prove that the sum $S_2(n)$ of the squares of the first $n$ natural numbers is equal to $\frac{n(n+1)(2n+1)}{6}$:

$$S_2(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (3)$$

*Solution.* 1°. $S_2(1) = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$.

2°. Assume that

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$S_2(n+1) = 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

and finally

$$S_2(n+1) = \frac{(n+1)[(n+1)+1)][2(n+1)+1]}{6}.$$

PROBLEM 1. Prove that the sum $S_3(n)$ of the cubes of the first $n$ natural numbers equals $\frac{n^2(n+1)^2}{4}$:

$$S_3(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}. \qquad (4)$$

EXAMPLE 4. Prove that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = \frac{(n-1)n(n+1)}{3}. \qquad (5)$$

*Solution.* 1°. $1 \times 2 = \frac{1 \times 2 \times 3}{3}$.

2°. If

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = \frac{(n-1)n(n+1)}{3},$$

then

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n + n(n+1) = \frac{(n-1)n(n+1)}{3} + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

PROBLEM 2. Deduce formula (5) from formulas (2) and (3).

*Hint.* First show that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = (1^2 + 2^2 + 3^2 + \ldots + n^2) - (1 + 2 + 3 + \ldots + n).$$

The method of mathematical induction, being in its essence connected with the notion of number, is most widely employed in arithmetic, algebra and the theory of numbers. Many interesting examples of this kind can be found in the booklet by I. S. Sominsky mentioned in the Preface to this book. But the notion of the whole number is a fundamental one, not only in the theory of numbers, whose subject is the whole numbers and their properties, but in all fields of mathematics. Therefore, the method of mathematical induction is used in many different branches of mathematics. But applications of this method in geometry are especially beautiful, and they form the subject matter of this book. The material is divided into six sections, each being dedicated to a particular type of geometric problem.

## Sec. 1. Calculation by Induction

The most natural use of the method of mathematical induction in geometry, one which is close to its use in the theory of numbers in algebra, is its application to solving computational problems in geometry.

EXAMPLE 5. Calculate the side $a_{2^n}$ of a regular $2^n$-gon inscribed in a circle of radius $R$.

*Solution.* 1°. For $n = 2$ the $2^n$-gon is a square; its side $a_4 = R\sqrt{2}$. Then according to the duplication formula

$$a_{2^{n+1}} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{a_{2^n}^2}{4}}}$$

we find that the side of a regular octagon $a_8 = R\sqrt{2-\sqrt{2}}$, the side of a regular 16-gon $a_{16} = R\sqrt{2-\sqrt{2+\sqrt{2}}}$, the side of a regular 32-gon $a_{32} = R\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$. We may therefore assume that for any $n \geqslant 2$ the side of a regular inscribed $2^n$-gon

$$a_{2^n} = R\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-2 \text{ times}}}. \qquad (6)$$

2°. Suppose that the side of a regular inscribed $2^n$-gon is expressed by formula (6). Then by the duplication formula

$$a_{2^{n+1}} = \sqrt{2R^2 - 2R\sqrt{R^2 - R^2\frac{2 - \overbrace{\sqrt{2 + \ldots + \sqrt{2}}}^{}_{n-2 \text{ times}}}{4}}} =$$

$$= R\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-1 \text{ times}}},$$

whence it follows that formula (6) is valid for all $n$.

It follows from formula (6) that the length of a circle of radius $R$ $(C = 2\pi R)$ is equal to the limit of the expression $2^n R\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-2 \text{ times}}}$ as $n$ increases unboundedly and hence

$$\pi = \lim_{n \to \infty} 2^{n-1}\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-2 \text{ times}}} = \lim_{n \to \infty} 2^n\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-1 \text{ times}}}.$$

PROBLEM 3. Using formula (6), prove that $\pi$ equals the limit to which the expression

$$\frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}}\right)}\sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}}\right)}\right)}\cdots}$$

tends as the number of factors (square roots) in the denominator increases unboundedly (Vieta's formula*). The way the factors are formed can be determined from the first three factors (which have been given).

* F. Vieta (1540 – 1603), a well-known French mathematician, one of the creators of algebraic symbols.