LIE GROUPS, LIE ALGEBRAS, *and their* REPRESENTATIONS

V. S. VARADARAJAN

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PRENTICE-HALL, INC.

Englewood Cliffs, N.J.

Library of Congress Cataloging in Publication Data

VARADARAJAN, V. S.
Lie groups, Lie algebras, and their representations.

Bibliography: p.
1. Lie groups. 2. Lie algebras. 3. Representations of groups. I. Title.
QA387.V35 512'.55 74-5441
ISBN 0-13-535732-2

10 9 8 7 6 5 4 3 2 1

Printed in the United States of America.

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

PREFACE

ओं तत्सवितुर्वरेण्यम् भर्गो देवस्य धीमहि धियो यो नः प्रचोदयात् ॥

This book has grown out of a set of lecture notes I had prepared for a course on Lie groups in 1966. When I lectured again on the subject in 1972, I revised the notes substantially. It is the revised version that is now appearing in book form.

The theory of Lie groups plays a fundamental role in many areas of mathematics. There are a number of books on the subject currently available —most notably those of Chevalley, Jacobson, and Bourbaki—which present various aspects of the theory in great depth. However, I feel there is a need for a single book in English which develops both the algebraic and analytic aspects of the theory and which goes into the representation theory of semi-simple Lie groups and Lie algebras in detail. This book is an attempt to fill this need. It is my hope that this book will introduce the aspiring graduate student as well as the nonspecialist mathematician to the fundamental themes of the subject.

I have made no attempt to discuss infinite-dimensional representations. This is a very active field, and a proper treatment of it would require another volume (if not more) of this size. However, the reader who wants to take up this theory will find that this book prepares him reasonably well for that task.

I have included a large number of exercises. Many of these provide the reader opportunities to test his understanding. In addition I have made a systematic attempt in these exercises to develop many aspects of the subject that could not be treated in the text: homogeneous spaces and their cohomologies, structure of matrix groups, representations in polynomial rings, and complexifications of real groups, to mention a few. In each case the exercises are graded in the form of a succession of (locally simple, I hope) steps, with hints for many. Substantial parts of Chapters 2, 3 and 4, together with a suitable selection from the exercises, could conceivably form the content of a one year graduate course on Lie groups. From the student's point of view the prerequisites for such a course would be a one-semester course on topological groups and one on differentiable manifolds.

The book begins with an introductory chapter on differentiable and analytic manifolds. A Lie group is at the same time a group and a manifold, and the theory of differentiable manifolds is the foundation on which the subject should be built. It was not my intention to be exhaustive, but I have made an effort to treat the main results of manifold theory that are used subsequently, especially the construction of global solutions to involutive systems of differential equations on a manifold. In taking this approach I have followed Chevalley, whose Princeton book was the first to develop the theory of Lie groups globally. My debt to Chevalley is great not only here but throughout the book, and it will be visible to anyone who, like me, learned the subject from his books.

The second chapter deals with the general theory. All the basic results and concepts are discussed: Lie groups and their Lie algebras, the correspondence between subgroups and subalgebras, the exponential map, the Campbell–Hausdorff formula, the theorems known as the fundamental theorems of Lie, and so on.

The third chapter is almost entirely on Lie algebras. The aim is to examine the structure of a Lie algebra in detail. With the exception of the last part of this chapter, where applications are made to the structure of Lie groups, the action takes place over a field of characteristic zero. The main results are the theorems of Lie and Engel on nilpotent and solvable algebras; Cartan's criterion for semisimplicity, namely that a Lie algebra is semisimple if and only if its Cartan–Killing form is nonsingular; Weyl's theorem asserting that all finite-dimensional representations of a semisimple Lie algebra are semisimple; and the theorems of Levi and Mal'čev on the semidirect decompositions of an arbitrary Lie algebra into its radical and a (semisimple) Levi factor. Although the results of Weyl and Levi–Mal'čev are cohomological in their nature (at least from the algebraic point of view), I have resisted the temptation to discuss the general cohomology theory of Lie algebras and have confined myself strictly to what is needed (*ad hoc* low-dimensional cohomology).

The fourth and final chapter is the heart of the book and is a fairly complete treatment of the fine structure and representation theory of semisimple Lie algebras and Lie groups. The root structure and the classification of simple Lie algebras over the field of complex numbers are obtained. As for representation theory, it is examined from both the infinitesimal (Cartan, Weyl, Harish-Chandra, Chevalley) and the global (Weyl) points of view. First I present the algebraic view, in which universal enveloping algebras, left ideals, highest weights, and infinitesimal characters are put in the foreground. I have followed here the treatment of Harish-Chandra given in his early papers and used it to prove the bijective nature of the correspondence between connected Dynkin diagrams and simple Lie algebras over the complexes. This algebraic part is then followed up with the transcendental theory. Here compact Lie groups come to the fore. The existence and conjugacy of their maximal tori are established, and Weyl's classic derivation of his great character formula is given. It is my belief that this dual treatment of representation theory is not only illuminating but even essential and that the infinitesimal and global parts of the theory are complementary facets of a very beautiful and complete picture.

In order not to interrupt the main flow of exposition, I have added an appendix at the end of this chapter where I have discussed the basic results of finite reflection groups and root systems. This appendix is essentially the same as a set of unpublished notes of Professor Robert Steinberg on the subject, and I am very grateful to him for allowing me to use his manuscript.

It only remains to thank all those without whose help this book would have been impossible. I am especially grateful to Professor I. M. Singer for his help at various critical stages. Mrs. Alice Hume typed the entire manuscript, and I cannot describe my indebtedness to the great skill, tempered with great patience, with which she carried out this task. I would like to thank Joel Zeitlin, who helped me prepare the original 1966 notes; and Mohsen Pazirandeh and Peter Trombi, who looked through the entire manuscript and corrected many errors. I would also like to thank Ms. Judy Burke, whose guidance was indispensable in preparing the manuscript for publication.

I would like to end this on a personal note. My first introduction to serious mathematics was from the papers of Harish-Chandra on semisimple Lie groups, and almost everything I know of representation theory goes back either to his papers or the discussions I have had with him over the past years. My debt to him is too immense to be detailed.

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CONTENTS

CHAPTER 1

DIFFERENTIABLE AND ANALYTIC MANIFOLDS

1.1. Differentiable Manifolds

We shall devote this chapter to a summary of those concepts and results from the theory of differentiable and analytic manifolds which are needed for our work in the rest of the book. Most of these results are standard and adequately treated in many books (see for example Chevalley [1], Helgason [1], Kobayashi and Nomizu [1], Bishop and Crittenden [1], Narasimhan [1]).

Differentiable structures. For technical reasons we shall permit our differentiable manifolds to have more than one connected component. However, all the manifolds that we shall encounter are assumed to satisfy the second axiom of countability and to have the same dimension at all points. More precisely, let M be a Hausdorff topological space satisfying the second axiom of countability. By a (C^∞) *differentiable structure* on M we mean an assignment

$$\mathfrak{D} : U \mapsto \mathfrak{D}(U) \quad (U \text{ open}, \subseteq M)$$

with the following properties:

(i) for each open $U \subseteq M$, $\mathfrak{D}(U)$ is an algebra of complex-valued functions on U containing 1 (the function identically equal to unity)

(ii) if V, U are open, $V \subseteq U$ and $f \in \mathfrak{D}(U)$, then $f|V \in \mathfrak{D}(V)$;[1] if V_i ($i \in J$) are open, $V = \cup_i V_i$, and f is a complex-valued function defined on V such that $f|V_i \in \mathfrak{D}(V_i)$ for all $i \in J$, then $f \in \mathfrak{D}(V)$

(iii) there exists an integer $m > 0$ with the following property: for any $x \in M$, one can find an open set U containing x, and m *real* functions $x_1, \ldots, x_m$ from $\mathfrak{D}(U)$ such that (a) the map

$$\xi : y \mapsto (x_1(y), \ldots, x_m(y))$$

is a homeomorphism of U onto an open subset of $\mathbf{R}^m$ (real m-space), and (b)

[1]If F is any function defined on a set A, and $B \subseteq A$, then $F|B$ denotes the restriction of F to B.

for any open set $V \subseteq U$ and any complex-valued function f defined on V, $f \in \mathfrak{D}(V)$ if and only if $f \circ \xi^{-1}$ is a C^∞ function on $\xi[V]$.

Any open set U for which there exist functions $x_1, \ldots, x_m$ having the property described in (iii) is called a *coordinate patch;* $\{x_1, \ldots, x_m\}$ is called a *system of coordinates on U*. Note that for any open $U \subseteq M$, the elements of $\mathfrak{D}(U)$ are continuous on U.

It is not required that M be connected; it is, however, obviously locally connected and metrizable. The integer m in (iii) above, which is the same for all points of M, is called the *dimension* of M. The pair $(M, \mathfrak{D})$ is called *differentiable* (C^∞) *manifold.* By abuse of language, we shall often refer to M itself as a differentiable manifold. It is usual to write $C^\infty(U)$ instead of $\mathfrak{D}(U)$ for any open set $U \subseteq M$ and to refer to its elements as (C^∞) *differentiable functions* on U. If U is any open subset of M, the assignment $V \mapsto C^\infty(V)$ ($V \subseteq U$, open) gives a C^∞ structure on U. U, equipped with this structure, is a C^∞ manifold having the same dimension as M; it is called the *open submanifold* defined by U. The connected components of M are all open submanifolds of M, and there can be at most countably many of these.

Let k be an integer ≥ 0, $U \subseteq M$ any open set. A complex-valued function f defined on U is said to be of *class* C^k on U if, around each point of U, f is a k-times continuously differentiable function of the local coordinates. It is easy to see that this property is independent of the particular set of local coordinates used. The set of all such f is denoted by $C^k(U)$. (We omit k when $k = 0$: $C(U) = C^0(U)$. $C^k(U)$ is an algebra over the field of complex numbers **C** and contains $C^\infty(U)$.

Given any complex-valued function f on M, its *support*, supp f, is defined as the complement in M of the largest open set on which f is identically zero. For any open set U and any integer k with $0 \leq k \leq \infty$, we denote by $C_c^k(U)$ the subspace of all $f \in C^k(M)$ for which supp f is a compact subset of U.

There is no difficulty in constructing nontrivial elements of $C^\infty(M)$. We mention the following results, which are often useful.

(i) Let $U \subseteq M$ be open and $K \subseteq U$ be compact; then we can find $\varphi \in C^\infty(M)$ such that $0 \leq \varphi(x) \leq 1$ for all x, with $\varphi = 1$ in an open set containing K, and $\varphi = 0$ outside U.

(ii) Let $\{V_i\}_{i \in J}$ be a locally finite[2] open covering of M with $Cl(V_i)$ (Cl denoting closure) compact for all $i \in J$; then there are $\varphi_i \in C^\infty(M)(i \in J)$ such that

(a) for each $i \in J$ $\varphi_i \geq 0$ and supp φ_i is a (compact) subset of V_i

(b) $\sum_{i \in J} \varphi_i(x) =$ for all $x \in M$ (this is a finite sum for each x, since $\{V_i\}_{i \in J}$ is locally finite).

$\{\varphi_i\}_{i \in J}$ is called a *partition of unity subordinate to the covering* $\{V_i\}_{i \in J}$.

[2]A family $\{E_i\}_{i \in J}$ of subsets of a topological space S is called *locally finite* if each point of X has an open neighborhood which meets E_i for only finitely many $i \in J$.

Tangent vectors and differential expressions. Let M be a C^∞ manifold of dimension m, fixed throughout the rest of this section. Let $x \in M$. Two C^∞ functions defined around x are called *equivalent if* they coincide on an open set containing x. The equivalence classes corresponding to this relation are known as *germs of C^∞ functions at x*. For any C^∞ function f defined around x we write $\mathbf{f}_x$ for the corresponding germ at x. The algebraic operations on the set of differentiable functions give rise in a natural and obvious fashion to algebraic operations on the set of germs at x, converting the latter into an algebra over $\mathbf{C}$; we denote this algebra by $\mathbf{D}_x$. A germ is called *real* if it is defined by a real C^∞ function. The real germs form an algebra over $\mathbf{R}$. For any germ $\mathbf{f}$ at x we write $\mathbf{f}(x)$ to denote the common value at x of all the C^∞ functions belonging to $\mathbf{f}$. It is easily seen that any germ at x is determined by a C^∞ function defined on all of M.

Let $\mathbf{D}_x^*$ be the algebraic dual of the complex vector space $\mathbf{D}_x$, i.e., the complex vector space of all linear maps of $\mathbf{D}_x$ into $\mathbf{C}$. An element of $\mathbf{D}_x^*$ is said to be *real* if it is real-valued on the set of real germs. A *tangent vector to M at x* is an element v of $\mathbf{D}_x^*$ such that

$$(1.1.1)\qquad \begin{cases} \text{(i)} & v \text{ is real} \\ \text{(ii)} & v(\mathbf{fg}) = \mathbf{f}(x)v(\mathbf{g}) + \mathbf{g}(x)v(\mathbf{f}) \text{ for all } \mathbf{f}, \mathbf{g} \in \mathbf{D}_x. \end{cases}$$

The set of all tangent vectors to M at x is an $\mathbf{R}$-linear subspace of $\mathbf{D}_x^*$, and is denoted by $T_x(M)$; it is called the *tangent space to M at x*. Its complex linear span $T_{xc}(M)$ is the set of all elements of $\mathbf{D}_x^*$ satisfying (ii) of (1.1.1). Let U be a coordinate patch containing x with $x_1, \ldots, x_m$ a system of coordinates on U, and let

$$\tilde{U} = \{(x_1(y), \ldots, x_m(y)) : y \in U\}.$$

For any $f \in C^\infty(U)$ let $\tilde{f} \in C^\infty(\tilde{U})$ be such that $\tilde{f} \circ (x_1, \ldots, x_m) = f$. Then the maps

$$f \mapsto \left(\frac{\partial \tilde{f}}{\partial t_j}\right)_{t_1 = x_1(x), \ldots, t_m = x_m(x)}$$

for $1 \leq j \leq m$ ($t_1, \ldots, t_m$ being the usual coordinates on $\mathbf{R}^m$) induce linear maps of $\mathbf{D}_x$ into $\mathbf{C}$ which are easily seen to be tangent vectors; we denote these by $(\partial/\partial x_j)_x$. They form a basis for $T_x(M)$ over $\mathbf{R}$ and hence of $T_{xc}(M)$ over $\mathbf{C}$.

Define the element $1_x \in \mathbf{D}_x^*$ by

$$(1.1.2)\qquad 1_x(\mathbf{f}) = \mathbf{f}(x) \quad (\mathbf{f} \in \mathbf{D}_x).$$

1_x is real and linearly independent of $T_x(M)$. It is easy to see that for an element $v \in \mathbf{D}_x^*$ to belong to the complex linear span of 1_x and $T_x(M)$ it is necessary and sufficient that $v(\mathbf{f}_1 \mathbf{f}_2) = 0$ for all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{D}_x$ which vanish at x. This leads naturally to the following generalization of the concept of a tangent vector. Let

$$\mathbf{J}_x = \{\mathbf{f} : \mathbf{f} \in \mathbf{D}_x, \mathbf{f}(x) = 0\} \tag{1.1.3}$$

Then $\mathbf{J}_x$ is an ideal in $\mathbf{D}_x$. For any integer $p \geq 1$, $\mathbf{J}_x^p$ is defined to be the linear span of all elements which are products of p elements from $\mathbf{J}_x$; $\mathbf{J}_x^p$ is also an ideal in $\mathbf{D}_x$. For any integer $r \geq 0$ we define a *differential expression of order* $\leq r$ to be any element of $\mathbf{D}_x^*$ which vanishes on $\mathbf{J}_x^{r+1}$; the set of all such is a linear subspace of $\mathbf{D}_x^*$ and is denoted by $T_{xc}^{(r)}(M)$. The real elements in $T_{xc}^{(r)}(M)$ from an **R**-linear subspace of $T_{xc}^{(r)}(M)$, spanning it (over **C**), and is denoted by $T_x^{(r)}(M)$. We have $T_x^{(0)}(M) = \mathbf{R}\cdot 1_x$, $T_x^{(1)}(M) = \mathbf{R}\cdot 1_x + T_x(M)$, and $T_x^{(r)}(M)$ increases with increasing r. Put

$$\begin{aligned} T_x^{(\infty)}(M) &= \bigcup_{r\geq 0} T_x^{(r)}(M) \\ T_{xc}^{(\infty)}(M) &= \bigcup_{r\geq 0} T_{xc}^{(r)}(M). \end{aligned} \tag{1.1.4}$$

$T_{xc}^{(\infty)}(M)$ is a linear subspace of $\mathbf{D}_x^*$, and $T_x^{(\infty)}(M)$ is an **R**-linear subspace spanning it over **C**.

It is easy to construct natural bases of the $T_x^{(r)}(M)$ in local coordinates. Let U be a coordinate patch containing x and let $\tilde{U}$ and $x_1, \ldots, x_m$ be as in the discussion concerning tangent vectors. Let (α) be any multiindex, i.e., $(\alpha) = (\alpha_1, \ldots, \alpha_m)$ where the α_j are integers ≥ 0; put $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Then the map

$$f \mapsto \left(\frac{\partial^{|\alpha|}\tilde{f}}{\partial t_1^{\alpha_1}\cdots\partial t_m^{\alpha_m}}\right)_{t_1=x_1(x), t_m=x_m(x)} \qquad (f \in C^\infty(U))$$

induces a linear function on $\mathbf{D}_x$ which is real. Let $\partial_x^{(\alpha)}$ denote this (when $(\alpha) = (0)$, $\partial_x^{(\alpha)} = 1_x$). Clearly, $\partial_x^{(\alpha)} \in T_x^{(r)}(M)$ if $|\alpha| \leq r$.

Lemma 1.1.1. *Let $r \geq 0$ be an integer and let $x \in M$. Then the differential expressions $\partial_x^{(\alpha)}$ ($|\alpha| \leq r$) form a basis for $T_x^{(r)}(M)$ over* **R** *and for $T_{xc}^{(r)}(M)$ over* **C**.

Proof. Since this is a purely local result, we may assume that M is the open cube $\{(y_1, \ldots, y_m) : |y_j| < a \text{ for } 1 \leq j \leq m\}$ in $\mathbf{R}^m$ with x as the origin. Let $t_1, \ldots, t_m$ be the usual coordinates, and for any multiindex $(\beta) = (\beta_1, \ldots, \beta_m)$ let $\mathbf{t}^{(\beta)}$ denote the germ at the origin defined by $t_1^{\beta_1} \ldots t_m^{\beta_m}/\beta_1! \cdots \beta_m!$

Let f be a real C^∞ function on M and let $g_{x_1,\ldots,x_m}(t) = f(tx_1, \ldots, tx_m)$ $(-1 \leq t \leq 1, (x_1, \ldots, x_m) \in M)$. By expanding $g_{x_1,\ldots,x_m}$ about $t = 0$ in its Taylor series, we get

$$g_{x_1,\ldots,x_m}(t) = \sum_{0\leq s\leq r} \frac{t^s}{s!} g_{x_1,\ldots,x_m}^{(s)}(0) + \frac{1}{r!}\int_0^t (t-u)^r g_{x_1,\ldots,x_m}^{(r+1)}(u)\,du$$

for $0 \leq t \leq 1$. Putting $t = 1$ and evaluating the t-derivatives of $g_{x_1,\ldots,x_m}$ in terms of the partial derivatives of f, we get, for all $(x_1, \ldots, x_m) \in M$,

$$f(x_1, \ldots, x_m) = \sum_{|\beta| \leq r} \frac{x_1^{\beta_1} \cdots x_m^{\beta_m}}{\beta_1! \cdots \beta_m!} \partial_x^{(\beta)}(\mathbf{f}) + \sum_{|\alpha| = r+1} \frac{x_1^{\alpha_1} \cdots x_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!} h^{(\alpha)}(x_1, \ldots, x_m),$$

where

$$h^{(\alpha)}(x_1, \ldots, x_m) = (r + 1) \int_0^1 (1 - u)^r \left(\frac{\partial^{\alpha_1 + \cdots + \alpha_m} f}{\partial t_1^{\alpha_1} \cdots \partial t_m^{\alpha_m}}\right)(ux_1, \ldots, ux_m)\, du.$$

Clearly, the $h^{(\alpha)}$ are real C^∞ functions on M. Passing to the germs at the origin, we get

$$\mathbf{f} = \sum_{|\beta| \leq r} \partial_x^{(\beta)}(f)\mathbf{t}^{(\beta)} + \sum_{|\alpha| = r+1} \mathbf{t}^{(\alpha)}\mathbf{h}^{(\alpha)}.$$

Since $\mathbf{t}^{(\alpha)} \in \mathbf{J}_x^{r+1}$ for any (α) with $|\alpha| = r + 1$, we get, for any $\lambda \in T_x^{(r)}(M)$,

$$\lambda = \sum_{|\beta| \leq r} \lambda(\mathbf{t}^{(\beta)})\partial_x^{(\beta)}$$

This shows that the $\partial_x^{(\beta)}(|\beta| \leq r)$ span $T_x^{(r)}(M)$ over $\mathbf{R}$. On the other hand, the $\partial_x^{(\beta)}$ are linearly independent over $\mathbf{R}$ or $\mathbf{C}$, since

$$\partial_x^{(\beta)}(\mathbf{t}^{(\gamma)}) = \begin{cases} 0 & (\gamma) \neq (\beta) \\ 1 & (\gamma) = (\beta) \end{cases}$$

This proves the lemma.

Vector fields. Let $X\,(x \mapsto X_x)$ be any assignment such that $X_x \in T_{xc}(M)$ for all $x \in M$. Then for any function $f \in C^\infty(M)$, the function $Xf\colon x \mapsto X_x(\mathbf{f}_x)$ is well defined on M, $\mathbf{f}_x$ being the germ at x defined by f. If U is any coordinate patch and $x_1, \ldots, x_m$ are coordinates on U, there are unique complex-valued functions $a_1, \ldots, a_m$ on U such that

$$X_y = \sum_{1 \leq j \leq m} a_j(y)\left(\frac{\partial}{\partial x_j}\right)_y \quad (y \in U).$$

X is called a *vector field on* M if $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$, or equivalently, if for each $x \in M$ there exist a coordinate patch U containing x and coordinates $x_1, \ldots, x_m$ on U such that the a_j defined above are C^∞ functions on U. A vector field X is said to be *real* if $X_x \in T_x(M)\ \forall\ x \in M$; X is real if and only if Xf is real for all real $f \in C^\infty(M)$. Given a vector field X, the mapping $f \longrightarrow Xf$ is a derivation of the algebra $C^\infty(M)$; i.e., for all f and

$g \in C^\infty(M)$,

(1.1.5) $$X(fg) = f \cdot Xg + g \cdot Xf.$$

This correspondence between vector fields and derivations is one to one and maps the set of all vector fields onto the set of all derivations of $C^\infty(M)$. Denote by $\mathfrak{Z}(M)$ the set of all vector fields on M. If $X \in \mathfrak{Z}(M)$ and $f \in C^\infty(M)$, $fX: x \mapsto f(x)X_x$ is also a vector field. In this way, $\mathfrak{Z}(M)$ becomes a module over $C^\infty(M)$. We make in general no distinction between a vector field and the corresponding derivation of $C^\infty(M)$.

Let X and Y be two vector fields. Then $X \circ Y - Y \circ X$ is an endomorphism of $C^\infty(M)$ which is easily verified to be a derivation. The associated vector field is denoted by $[X, Y]$ and is called the *Lie bracket* of X with Y. The map

$$(X, Y) \mapsto [X, Y]$$

is bilinear and possesses the following easily verified properties:

(1.1.6) $$\begin{cases} \text{(i)} & [X,X] = 0 \\ \text{(ii)} & [X,Y] + [Y,X] = 0 \\ \text{(iii)} & [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \end{cases}$$

(X, Y, and Z being arbitrary in $\mathfrak{Z}(M)$). If X and Y are real, so is $[X, Y]$. The relation (iii) of (1.1.6) is known as the *Jacobi identity*.

Differential operators. Let $r \geq 0$ be an integer and let

(1.1.7) $$D: x \mapsto D_x$$

be an assignment such that $D_x \in T^{(r)}_{xc}(M)$ for all $x \in M$. If $f \in C^\infty(M)$, the function $Df: x \mapsto D_x(\mathbf{f}_x)$ is well defined on M, $\mathbf{f}_x$ being the germ defined by f at x. If U is a coordinate patch and $x_1, \ldots, x_m$ are coordinates on U, then by Lemma 1.1.1 there are unique complex functions $a_{(\alpha)}$ on U such that

$$D_y = \sum_{|\alpha| \leq r} a_{(\alpha)}(y)\partial_y^{(\alpha)} \quad (y \in U).$$

D is called a *differential operator on M* if $Df \in C^\infty(M)$ for all $f \in C^\infty(M)$, or equivalently, if for each $x \in M$ we can find a coordinate patch U containing x with coordinate $x_1, \ldots, x_m$ such that the $a_{(\alpha)}$ defined above are in $C^\infty(U)$. The smallest integer $r \geq 0$ such that $D_x \in T^{(r)}_{xc}(M)$ for all $x \in M$ is called the *order* (ord(D)) or the *degree* (deg(D)) of D. For any differential operator D on M and $x \in M$, D_x is called the *expression* of D at x. If Df is real for any real-valued $f \in C^\infty(M)$, we say that D is *real*. The set of all differential operators on M is denoted by Diff(M). If $f \in C^\infty(M)$ and $D \in$ Diff(M), $fD: x \mapsto f(x)\, D_x$ is again a differential operator; its order cannot exceed the order of D. Thus Diff(M) is a module over $C^\infty(M)$. A *vector field* is a differential operator of order ≤ 1. If $\{V_i\}_{i \in J}$ is an open covering of M and $D_i(i \in J)$ is a differential operator on V_i such that

(a) $\sup_{i \in J} \operatorname{ord}(D_i) < \infty$
(b) if $V_{i_1} \cap V_{i_2} \cap \phi$, the restrictions of D_{i_1} and D_{i_2} to $V_{i_1} \cap V_{i_2}$ are equal,

then there exists exactly one differential operator D on M such that for any $i \in J$ D_i is the restriction of D to V_i.

Let D $(x \mapsto D_x)$ be a differential operator of order $\leq r$. We also denote by D the endomorphism $f \mapsto Df$ of $C^\infty(M)$. This endomorphism is then easily verified to have the following properties:

(1.1.8)
(i) it is *local;* i.e., if $f \in C^\infty(M)$ vanishes on an open set U, Df also vanishes on U
(ii) if $x \in M$, and $f_1, \ldots, f_{r+1}$ are $r + 1$ functions in $C^\infty(M)$ which vanish at x, then

$$(D(f_1 f_2 \cdots f_{r+1}))(x) = 0.$$

Conversely, it is quicky verified that given any endomorphism E of $C^\infty(M)$ satisfying (ii) of (1.1.8) for some integer $r \geq 0$, E is local and there is exactly one differential operator D on M such that $Df = Ef$ for all $f \in C^\infty(M)$; and $\operatorname{ord}(D) \leq r$. In view of this, we make no distinction between a differential operator and the endomorphism of $C^\infty(M)$ induced by it. It follows easily from the expression of a differential operator in local coordinates that if D_1 and D_2 are differential operators of respective orders r_1 and r_2, then $D_1 D_2$ is also a differential operator, and its order is $\leq r_1 + r_2$; moreover, $D_1 D_2 - D_2 D_1$ is a differential operator of order $\leq r_1 + r_2 - 1$. Diff(M) is thus an algebra (not commutative); if Diff$(M)_r$ is the set of elements of Diff(M) of order $\leq r$, $r \mapsto$ Diff$(M)_r$ converts Diff(M) into a filtered algebra. A differential operator of order 0 is just the operator of multiplication by a C^∞ function; if u is in $C^\infty(M)$ we denote again by u the operator $f \mapsto uf$ of $C^\infty(M)$.

If $M = \mathbf{R}^m$ and D is a differential operator of order $\leq r$, there are unique C^∞ functions $a_{(\alpha)}$ $(|\alpha| \leq r)$ on M (*coefficients* of D) such that

$$D = \sum_{|\alpha| \leq r} a_{(\alpha)} \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \cdots \partial t_m^{\alpha_m}},$$

$t_1, \ldots, t_m$ being the linear coordinates on M. It is natural to ask whether such global representations exist on more general manifolds. The following theorem gives one such result.

Theorem 1.1.2. *Let $X_1, \ldots, X_m$ be m vector fields on M such that $(X_1)_x, \ldots, (X_m)_x$ form a basis of $T_{xc}(M)$ for each $x \in M$. For any multiindex $(\alpha) = (\alpha_1, \ldots, \alpha_m)$ let $X^{(\alpha)}$ be the differential operator*

$$X^{(\alpha)} = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_m^{\alpha_m} \tag{1.1.9}$$

(when $(\alpha) = (0) X^{(\alpha)} = 1$, the identity operator). Then the $X^{(\alpha)}$ are linearly independent over $C^\infty(M)$. If D is any differential operator of order $\leq r$, we can find unique C^∞ functions $a_{(\alpha)}$ on M such that

$$D = \sum_{|\alpha| \leq r} a_{(\alpha)} X^{(\alpha)}. \tag{1.1.10}$$

If the X_i are real, then for any real differential operator D the $a_{(\alpha)}$ defined by (1.1.10) *are all real.*

Proof. For any integer $r \geq 0$, let $\mathfrak{D}_r$ denote the complex vector space of all differential operators on M of the form $\sum_{|\alpha| \leq r} f_{(\alpha)} X^{(\alpha)}$, the $f_{(\alpha)}$ being C^∞ functions on M. Note that $\mathfrak{D}_1$ contains all vector fields. In fact, if Z is any vector field, we can write $Z = \sum_{1 \leq j \leq m} c_j X_j$ for uniquely defined functions c_j. To see that the c_j are in $C^\infty(M)$, let U be a coordinate patch with coordinates $x_1, \ldots, x_m$. Then there are C^∞ functions d_j, a_{jk} on U $(1 \leq j, k \leq m)$ such that $Z_y = \sum_{1 \leq j \leq m} d_j(y)(\partial/\partial x_j)_y$ and $(X_j)_y = \sum_{1 \leq k \leq m} a_{jk}(y)(\partial/\partial x_k)_y$ for all $y \in U$. Since the $(X_j)_y$ $(1 \leq j \leq m)$ are linearly independent for all y, the matrix (a_{jk}) is invertible. If a^{jk} are the entries of the inverse matrix, they are in $C^\infty(U)$ and $c_j = \sum_{1 \leq j \leq m} d_k a^{kj}$ on U.

We begin the proof of the theorem by showing that if l is an integer ≥ 1 and $Z_1, \ldots, Z_l$ are l vector fields, then the product $Z_1 \cdots Z_l$ belongs to $\mathfrak{D}_l$. For $l = 1$, this is just the remark made in the previous paragraph. Proceed by induction on l. Let $l > 1$, and assume that the result holds for any $l - 1$ vector fields. Let $Z_1, \ldots, Z_l$ be l vector fields, and write $E = Z_1 \cdots Z_l$.

Notice first that if $Y_1, \ldots, Y_l$ are any l vector fields, $F = Y_1 \cdots Y_l$, and F' is the product obtained by intechanging two adjacent Y's, then $F - F'$ is a product of $l - 1$ vector fields. So $F - F' \in \mathfrak{D}_{l-1}$ by the induction hypothesis. Since any permutation is a product of such adjacent interchanges, it follows from the induction hypothesis that $Y_1 \cdots Y_l - Y_{i_1} Y_{i_2} \cdots Y_{i_l} \in \mathfrak{D}_{l-1}$ for any permutation $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$. But if $1 \leq j_1 \leq j_2 \leq \cdots \leq j_l \leq m$, then $X_{j_1} \cdots X_{j_l} = X^{(\alpha)}$ for a suitable (α) with $|\alpha| = l$, so that $X_{j_1} \cdots X_{j_l} \in \mathfrak{D}_l$. Hence, from what we proved above, if $(k_1, \ldots, k_m)$ is any permutation of $(1, \ldots, m)$ and (α) is any multi-index with $|\alpha| \leq l$, then $X_{k_1}^{\alpha_1} \cdots X_{k_m}^{\alpha_m} \in \mathfrak{D}_l$.