TOPOLOGICAL GROUPS

Characters, Dualities, and Minimal Group Topologies

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Preface

It would not be an exaggeration to say that the modern theory of topological groups depends strongly on functional analysis. Even the first deep theorem—concerning the existence of continuous characters on locally compact abelian groups—is based on the properties of symmetric compact operators on Hilbert spaces or on the theory of Banach algebras. This situation is not satisfactory in some sense—it would be desirable to construct at least the theory of locally compact topological groups directly, without recourse to "foreign" concepts. Our aim in this book is to use only those tools of functional analysis that belong to general topology: the Tichonov compactness theorem, the Stone-Weierstrass theorem, the Čech-Stone compactification, the Baire category theorem, and some others. We feel that our program has been successful in the case of topological abelian groups; we do not know whether a parallel program is available in the general case.

We rely heavily on a theorem of Erling Følner about the existence of continuous characters, which implies the Peter-Weyl theorem for locally compact abelian groups. From Følner's theorem one also obtains the existence of the Haar integral in locally compact abelian groups, as well as the theorem of Bohr-von Neumann concerning the approximation of almost periodic functions on abelian groups by linear combinations of characters.

One of the applications of Følner's theorem—minimal group topologies—is given particular emphasis, and this is in fact one of the main topics in this book. These are considered not only as applications of Følner's theorem but also for their connection with some problems in the areas of p-adic numbers and number theory, and of Lie groups, as well. The theory of minimal groups, developed over a period of more than 15 years, contains by now many interesting results, many of them described in the specialized surveys of Comfort and Grant (1981), Dikranjan (1983a), and Prodanov and Stoyanov (1983), as well as in some surveys dealing with topological groups in general [see, for example, Comfort (1984) and Arhangel'skii (1980, 1981, 1987)]. Nevertheless, this theory does not seem to be exhausted; new results are continually being obtained—13 of the 18 open problems included in Dikranjan (1983a) were solved in the last five years.

The aim of this book is to develop the theory of characters in locally compact abelian groups in the spirit described above, including Pontryagin duality and the structure theory of locally compact abelian groups, and to present most of the theory of minimal topological groups. A Hausdorff topological group G is called minimal if the topology of G is a minimal element (in Zorn's sense) of the set of all Hausdorff group topologies on G. Several intermediate results are given; some of them are of independent interest (e.g., the uniqueness of the Pontryagin duality). Examples include a new functorial topology for abelian groups, some classes of finite-dimensional compact abelian groups, and new cardinal invariants of compact abelian groups.

Several reasons made us concentrate mostly on the abelian case, but mainly it was for lack of a substitute for the Følner theorem in the abelian case. On the other hand, the theory of minimal groups is much better developed in the abelian case because of the Pontryagin duality and the precompactness of the minimal abelian groups. In the nonabelian case, examples and counterexamples prevail over the general theory. As an instance of the difference between the two cases, let us note that there exist non-precompact minimal (nonabelian) groups, and there exist minimal groups of arbitrary cardinality (a description of the cardinalities of the minimal abelian groups is given in Section 5.3). This difference is due to the nature of the minimal topologies—roughly speaking, "there are more minimal topologies in the nonabelian case." On a "strongly" nonabelian group there are a few Hausdorff group topologies, so there are more possibilities of having a minimal one. An extreme example is the well-known Adian's group A(m,n), which (as shown by Ol'shanskii) admits only the discrete topology as a Hausdorff group topology. To maintain a balance between the abelian and nonabelian cases, we collect in a separate chapter (Chapter 7) most of the results in this field of minimal groups.

The basic exposition is almost self-contained. The reader will need only some knowledge of general topology and basic algebra, mainly elements of group and module theory, within the framework of, say, Kelley (1959) and Fuchs (1970); the comments and supplements, however, are more broadly based. At the end of each chapter we provide various problems—from standard and easy exercises to open problems—and we give hints for most of them. This should make the book accessible to undergraduate students. In fact, the first three chapters could be used for an elementary course on topological groups theory at the undergraduate level.

Chapter 1 of this book contains a proof of Følner's theorem (1954a): If G is a topological abelian group and E is a subset of G such that k translates of E cover G, then for every neighborhood U of 0 in G there exist continuous characters $\chi_1, \ldots, \chi_{k^2}$ of G and a positive δ such that if $x \in G$ and $|\operatorname{Arg} \chi_i(x)| < \delta$ for $i = 1, \ldots, k^2$, then

$$x \in U - U + E - E + E - E + E - E + E - E$$

The proof presented here, due to Prodanov, is rather elementary; the deepest fact used is the Stone-Weierstrass theorem. This proof works, in fact, under less restrictive assumptions on G. Chapter 1 contains four sections; in the first we collect some basic properties of characters of discrete abelian groups. The proof of Følner's theorem is given in Section 1.4.

Chapter 2 contains some applications of Følner's theorem. It consists of eight sections, the first of which collects some necessary facts about topological groups. In Section 2.2 we deduce the Bohr-von Neumann theorem on uniform approximation of almost periodic functions in abelian groups by linear combinations of characters. In Section 2.3 a description (due to Comfort and Ross, 1964) is given of the precompact group topologies on an arbitrary abelian group G in terms of the groups of characters of G. The Peter-Weyl theorem for compact abelian groups is also included here. A construction of the Haar integral in locally compact abelian groups is contained in Section 2.4. The main result in Sections 2.5, 2.6, and 2.7 [proved in Prodanov and Stoyanov (1984) in a different manner] is that every minimal abelian group is precompact. The central fact here is Lemma 2.7.3, the proof of which makes essential use of Følner's theorem and which plays an important role in Section 2.7 as well. In fact, the authors do not know of any proof of the precompactness of minimal abelian groups that does not use Følner's theorem. Section 2.6 contains two criteria for minimality of a topological group G; the first of these (due to Bahaschewski, Prodanov, and Stephenson) relates the minimality of G to the minimality of its dense subgroups, while the second (due to Prodanov), in terms of the group of characters of G, is valid only in the abelian case. Finally, in Section 2.8 we give Prodanov's (1978) description of the submaximal topology of $\mathcal{M}_G$ of an arbitrary infinite abelian group G. Here $\mathcal{M}_G$ is the infimum, in the set of all group topologies on G, of all maximal nondiscrete group topologies on G.

Chapter 3 begins with a short proof of Pontryagin duality in the compact-discrete case based on the Peter-Weyl theorem and the minimality criterion proved in Chapter 2. In Section 3.2, Pontryagin duality is established for locally compact abelian groups. Here we follow the standard path which also leads to the structure theory of locally compact abelian groups (cf. Hewitt and Ross, 1963 and Morris, 1977). In Section 3.3 we show that up to equivalence the Pontryagin duality is the only abstract duality in the category of locally compact abelian groups. The proof of this fact, due to Prodanov (1982), is different from that of Roeder (1971). In fact, it permits us to establish uniqueness in the case of categories of locally compact modules over some discrete commutative rings R, although uniqueness is not always available, even in the case of a field R of characteristic zero (cf. Dikranjan, 1987).

In Sections 3.4 and 3.5 the duality theorem is applied to the study of the groups $\mathbb{Z}_p$ of p-adic integers and the dual group $\mathbb{Q}^*$ of the discrete additive group $\mathbb{Q}$ of the rational numbers. The group $\mathbb{Q}^*$ is used to provide with minimal group topologies the torsion-free abelian groups of infinite free rank not greater than $\mathfrak{c} = 2^{\aleph_0}$.

The main tool in the study of compact abelian groups in Chapter 4 and the following two chapters is a sort of localization which makes crucial use of the powerful p-adic technique. Roughly speaking, it associates with every compact abelian group G and every prime number p a subgroup $td_p(G)$ of G which is a $\mathbb{Z}_p$-module and contains the p-torsion part of G. On the other hand, the sum of these groups is large enough to carry all the necessary information about G. It turns out that the free rank m_p of $td_p(G)$ over $\mathbb{Z}_p$ is of great importance. When $m_p(G)$ is finite, the structure of $td_p(G)$ becomes rather transparent; in particular, its torsion part splits off. This allows us to describe in Section 4.2 the torsion parts of compact abelian groups G with finite $m_p(G)$ for all p.

A Hausdorff topological group is said to be <u>totally minimal</u> if all its Hausdorff quotients are minimal. In Section 4.3 we also give a criterion for total minimality of dense subgroups. Applying the localization technique, local criteria for minimality and total minimality of dense subgroups are also obtained.

It is shown in Section 5.1 that the compact abelian groups G with $m_p(G) \leq 1$ for every p are exactly the completions of the minimal abelian groups of free rank less than $\mathfrak{c}$. Those that have finite p-rank for every p are the completions of the countable minimal abelian groups. Finally, the compact abelian groups G with $m_p(G) = 0$ for all p are exactly the completions of the minimal torsion abelian groups. These characterizations and the structure theorem obtained in Section 4.2 enable us to give necessary conditions for the existence of minimal group topologies on abelian groups of free rank less than $\mathfrak{c}$. In the case of splitting groups these conditions also become sufficient. Various cases are considered. The divisible abelian groups admitting minimal group topologies are described, and in this way an extension of the well-known theorem of Hulanicki is obtained.

In Section 5.2 we characterize the groups of the p-adic integers as the only infinite complete abelian groups inducing minimal topologies on each subgroup. It is shown in Section 5.3 that if a compact abelian group contains a dense minimal subgroup of some cardinality σ, then it also contains a dense totally minimal subgroup of cardinality σ. The cardinalities of the minimal abelian groups are described.

In Chapter 6 we deal with products of minimal groups (note that in general these products are not minimal). In Section 6.1 we give a criterion for minimality of finite products and we characterize those minimal abelian groups, called perfectly minimal, for which the product with every minimal group is minimal. Here we also characterize the topological groups each power of which is (totally) minimal. In Section 6.2 we give a criterion for minimality of arbitrary products of minimal abelian groups. It reduces the question of minimality of an arbitrary product to the case when all groups in the product are subgroups of one and the same $\mathbb{Z}_p$. This permits us to show that the minimality of arbitrary products is determined by the minimality of subproducts of at most $\mathfrak{c}$ groups. We apply the criterion to obtain simpler conditions which guarantee the minimality of products in some concrete classes of minimal abelian groups.

In Section 6.3 we study the critical power of minimality. This is the smallest cardinal κ such that the minimality of G^{κ} implies the minimality of all powers of G. We show that $\aleph_0 < \kappa \leq \mathfrak{c}$; more precisely, $2^{\kappa} > \mathfrak{c}$. Therefore, under MA we have $\kappa = \mathfrak{c}$.

Chapter 7 contains most of the results on minimal (not necessarily abelian) groups known to us. Section 7.1 is devoted to a detailed study of the symmetric topological groups S(X). The main point is a result of Gaughan (1967) showing that the topology of pointwise convergence is the smallest (not merely minimal) Hausdorff group topology on S(X). The groups S(X) also served as the first examples of topological groups without completion (Theorem 7.1.3). This is due to Dieudonné (1944) and answers a question of Bourbaki (1966).

It turns out that the technique of semidirect products can be used successfully to find many examples of minimal topological groups. The first applications of this type were made by Dierolf and Schwanengel (1979), who established the existence of locally compact noncompact minimal groups. In Section 7.2 we consider some of their results, as well as an example, due to Schwanengel, of a precompact totally minimal group containing a nonminimal closed normal subgroup.

In Section 7.3 we discuss the so-called three-space problem: If N is a closed normal subgroup of G and both N and G/N are (totally) minimal, when is G (totally) minimal too? This problem was studied in detail by Eberhardt, Dierolf, and Schwanengel, and the first part of Section 7.3 is devoted to their results. The second part of this section deals mainly with the results of the same authors about total minimality of infinite products.

All examples of minimal locally compact groups given in Sections 7.2 and 7.3 are not totally minimal. In Section 7.4 we present a recent result of Remus and Stoyanov (1988) concerning total minimality of $SL(2, \mathbb{R})$, thus providing an example of a locally compact noncompact group all powers of which are totally minimal.

In Section 7.5 a theorem of Arhangel'skii is given about the coincidence of the weight and the net weight of a minimal group. The problem of Arhangel'skii about the coincidence of the character and the pseudocharacter of a minimal group is discussed next. A counterexample, due to Schakhmatov, is given showing that the gap between the character and the pseudocharacter of a minimal group may be arbitrarily large. In this direction we also give positive results, due to Comfort and Grant (1984) and Guran (1981b), for minimal groups satisfying some additional conditions.

Finally, in Section 7.6 we study the unitary groups $\mathfrak{U}(\mathcal{H})$ of the infinite-dimensional Hilbert spaces $\mathcal{H}$. The main result here, due to Stoyanov (1984), is that these groups are totally minimal. As a by-product we obtain some other properties of the groups $\mathfrak{U}(\mathcal{H})$.

During the preparation of this book we lost our senior coauthor and teacher, Professor Ivan Prodanov. He taught us functional analysis and introduced us to topological groups. Professor Prodanov was an excellent lecturer and a brilliant mathematician. His ground-breaking results on minimal groups and his original approach to Pontryagin duality form the core of this book. His personality and his work will always be remembered by everybody who knew him.

It is a pleasure to acknowledge our indebtedness to our colleagues W. Comfort, H. Herrlich, A. Orsatti, and L. Salce.

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Dikran N. Dikranjan
Luchezar N. Stoyanov

Contents

1

Existence of Characters

The main result in this chapter is the Følner theorem on the existence of continuous characters. Instead of topological abelian groups, as in the original result of Følner, we consider more general objects—abelian groups endowed with Hausdorff topologies with respect to which translations and multiplications by integers are continuous. In fact, the main exposition of this chapter does not require the notion of a topological group. On the other hand, the supplementary material in Section 5 uses some facts about topological groups. The reader unfamiliar with the basic material on topological groups should consult Section 2.1 and the references cited therein.

1.1 CHARACTERS OF DISCRETE ABELIAN GROUPS

Let G_1 and G_2 be abelian groups; a homomorphism from G_1 to G_2 is any map $h: G_1 \to G_2$ satisfying

$$h(x + y) = h(x) + h(y)$$

for arbitrary $x, y \in G_1$. We omit here the obvious modifications for the multiplicative notation.

The set $\mathbb{S}$ of complex numbers z with $|z| = 1$ is an abelian group with respect to multiplication of complex numbers. It is called the unit circle and plays a fundamental role in the theory of topological groups. The operation in this group will always be written multiplicatively. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the additive groups of reals and integers, respectively; then the quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{S}$, although it is not always possible to replace $\mathbb{S}$ by the additive group $\mathbb{T}$. It will become clear later in the chapter that $\mathbb{S}$ with its multiplicative notation fits perfectly in the proof of the main result of this chapter. On the other hand, in other cases—for example, in Pontryagin duality—it is convenient to replace $\mathbb{S}$ by $\mathbb{T}$.

For an abelian group G the group Hom(G, $\mathbb{S}$) will be called the (multiplicative) character group of G and its elements, characters of G. In other words, a character of G is a function $\chi: G \to \mathbb{S}$ satisfying $\chi(x + y) = \chi(x)\chi(y)$ for arbitrary $x, y \in G$. In such a case clearly $\chi(x - y) = \chi(x)\overline{\chi(y)}$ also holds, where $\bar{z}$ denotes complex conjugation. The multiplicative character group of G will be denoted by G'.

The following lemma will be used repeatedly in the sequel. It ensures the existence of a universal subnet in some sense.

1.1.1 LEMMA Let G be an abelian group and let $S = \{\chi_\alpha\}_\alpha$ be a net in G'. Then there exist a character χ of G and a subnet $\{\chi_{\alpha_\beta}\}_\beta$ of S such that $\chi_{\alpha_\beta}(x)$ converges in $\mathbb{S}$ to $\chi(x)$ for every $x \in G$.

Proof: The space $\mathbb{S}^G$ of all maps $G \to \mathbb{S}$ provided with the topology of pointwise convergence (i.e., the product topology on $\mathbb{S}^G$) is compact by Tichonov's theorem. Hence the net S in this space has a convergent subnet $\{\chi_{\alpha_\beta}\}_\beta$. Denote by χ its limit point; then $\chi_{\alpha_\beta}(x)$ converges to $\chi(x)$ for every $x \in G$. On the other hand, it is easy to see that G' is closed in S^G. So $\chi \in G'$. □

In case G is countable, the space $\mathbb{S}^G$ is metrizable; thus every sequence admits a convergent subsequence.

A fundamental problem in the theory of topological abelian groups is the construction of characters having appropriate continuity properties. The next theorem resolves this problem in the discrete case. We give it in more general form with a view to some applications in the following chapters.

Let us recall some notation. If G is an abelian group, $n \in \mathbb{Z}$, and A and B are subsets of G, then by definition

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and

$$nA = \{na : a \in A\}.$$

In the same way one can also define more complicated expressions, such as $A - B + C$ and $A + nB$. Similar notations can be introduced in the multiplicative case. We are not going to give the obvious rules to deal with such expressions.

Let p be a prime number. An abelian group H is said to be <u>divisible</u> (<u>p-divisible</u>) if for every natural number n we have $nH = H$ (respectively, $p^nH = H$). For example, the group $\mathbb{S}$ is divisible.

1.1.2 THEOREM Let G and H be abelian groups and let H be divisible. Then every homomorphism $h_0 : G_0 \to H$, where G_0 is an arbitrary subgroup of G, can be extended to a homomorphism $h: G \to H$. Moreover, if $a \in G\backslash G_0$ and H contains torsion elements of arbitrary orders, then h can be chosen in such a way that $h(a) \neq 0$.

Proof: Let G_1 be the subgroup of G generated by G_0 and a. Clearly, G_1 consists of all possible sums $g + na$, where $g \in G_0$ and $n \in \mathbb{Z}$. Two cases are possible:

(i) There exists a positive integer n with $na \in G_0$.
(ii) $na \notin G_0$ for every integer $n \neq 0$.

In case (i) denote by m the least positive integer n with $na \in G_0$. Then for every integer n with $na \in G_0$, m divides n. Since $a \notin G_0$, $m > 1$.

Choose an element $y \in H$ with $my = h_0(ma)$. This is possible since H is divisible. Moreover, if H has elements of arbitrary orders, then we may assume that $y \neq 0$. Now for $g + na \in G_1$ define

$$h_1(g + na) = h_0(g) + ny.$$

To establish the correctness of this definition, consider two representations $g_1 + n_1 a = g_2 + n_2 a$ of an element of G. By the choice of m, $(n_2 - n_1)a = g_1 - g_2 \in G$ implies that m divides $n_2 - n_1$. Let $n_2 - n_1 = km$; then

$$(n_2 - n_1)y = kh_0(ma) = h_0(n_2 a - n_1 a),$$

since h_0 is a homomorphism. The equalities

$$h_0(g_2) + n_2 y = h_0(g_2) + h_0(n_2 a - n_1 a) + n_1 y$$
$$= h_0(g_2 + n_2 a - n_1 a) + n_1 y = h_0(g_1) + n_1 y$$

show that the definition of h_1 is correct. It follows immediately that the map $h_1 : G_1 \to H$ obtained in this way is a homomorphism. If H contains torsion elements of arbitrary orders, then $h_1(a) = y \neq 0$.

In case (ii) choose an arbitrary nonzero element y of H. Defining h_1 as before, we get a homomorphism from G_1 into H. In this case the correctness of the definition of h_1 follows by the fact that every element of G_1 has a unique representation as a sum $g + na$ with $g \in G_0$ and $n \in \mathbb{Z}$.

In this way the homomorphism h_0 was extended over a larger subgroup G_1. If G is generated by G_0 and a countable subset of G, the proof can be finished by induction using the construction of h_1. In the general case one applies Zorn's lemma. □

1.1.3 COROLLARY Let G be an abelian group, G_0 be a subgroup of G, χ_0 be a character of G_0, and $a \in G \setminus G_0$. Then χ_0 can be extended to a character χ of G with $\chi(a) \neq 1$. □

For an abelian group G and a family Φ of functions on G we say that Φ separates the points of G if for each pair of distinct elements g_1 and g_2 of G there exists $f \in \Phi$ such that $f(g_1) \neq f(g_2)$.

1.1.4 COROLLARY The characters of every abelian group G separate the points of G. □

1.2 BOGOLIOUBOFF LEMMA

Here and in the sequel we denote by $|X|$ the cardinality of the set X. In this section G will denote a finite abelian group. The next proposition gives some useful properties of the characters of G.